Module-1

Set Theory

Basic Concepts of Set Theory: Symbols & Terminology

A **set** is a collection of objects.

A **well-defined set** has no ambiguity as to what objects are in the set or not.

For example:

- The collection of all red cars
- The collection of positive numbers
- The collection of people born before 1980
- The collection of greatest baseball players

All of these collections are sets. However, the collection of greatest baseball players is not well-defined.

Normally we restrict our attention to just well-defined sets.

Defining Sets

- **word description**

  The set of odd counting numbers between 2 and 12

- the **listing method**

  $\{3, 5, 7, 9, 11\}$

- **set-builder notation** or **defining property method**

  $\{x \mid x \text{ is a counting number, } x \text{ is odd, and } x < 12\}$

Note:

- Use curly braces to designate sets,
- Use commas to separate set elements
- The variable in the set–builder notation doesn't have to be x.
- Use ellipses $(\ldots)$ to indicate a continuation of a pattern established before the ellipses i.e. $\{1, 2, 3, 4, \ldots, 100\}$
- The symbol | is read as "such that"

Set Membership

An **element** or **member** of a set is an object that belongs to the set

The symbol $\in$ means "is an element of"

The symbol $\notin$ means "is not an element of"

Generally capital letters are used to represent sets and lowercase letters are used for other objects i.e. $S = \{2, 3, 5, 7\}$

Thus, $a \in S$ means a is an element of S

Is $2 \in \{0, 2, 4, 6\}$?

Is $2 \in \{1, 3, 5, 7, 9\}$?

Some Important Sets

- $\mathbb{N}$ — Natural or Counting numbers: {1, 2, 3, . . . }
- $\mathbb{W}$ — Whole Numbers: {0, 1, 2, 3, . . . }
- $\mathbb{I}$ — Integers: {. . . , -3, -2, -1, 0, 1, 2, 3, . . . }
- $\mathbb{Q}$ — Rational numbers: $\{\frac{p}{q} \mid p, q \in \mathbb{I}, q \neq 0\}$
- $\Re$ — Real Numbers: $\{ x \mid x$ is a number that can be written as a decimal $\}$
- Irrational numbers: $\{ x \mid x$ is a real number and x cannot be written as a quotient of integers $\}$.

  Examples are: π, $\sqrt{2}$, and $\sqrt[3]{4}$
- $\emptyset$ — Empty Set: { }, the set that contains nothing
- U — Universal Set: the set of all objects **currently** under discussion

Notes

Any **rational** number can be written as either a

terminating *decimal (like 0.5, 0.333, or 0.8578966)*
or a
repeating *decimal (like* $0.\overline{333}$ *or* $123.3925\overline{45}$*)*

The decimal representation of an **irrational** number **never terminates** and **never repeats**

The set $\{ \emptyset \}$ is *not* empty, but is a set which *contains* the empty set

More Membership Questions

- Is $\emptyset \in \{ a, b, c \}$?
- Is $\emptyset \in \{ \emptyset, \{ \emptyset \} \}$?
- Is $\emptyset \in \{ \{ \emptyset \} \}$?
- Is $\frac{1}{3} \notin \{ x \mid x = \frac{1}{p}, p \in \mathbb{N}\}$

Set Cardinality

The **cardinality** of a set is the number of **distinct** elements in the set

- The cardinality of a set A is denoted n(A) or $|A|$
- If the cardinality of a set is a particular whole number, we call that set a **finite** set
- If a set is so large that there is no such number, it is called an **infinite** set (there is a precise definition of infinity but that is beyond the scope of this course)

Note: Sets do not care about the order or how many times an object is included. Thus, $\{1, 2, 3, 4\}$, $\{2, 3, 1, 4\}$, and $\{1, 2, 2, 3, 3, 3, 4, 4\}$ all describe the same set.

Set Cardinality

$$A = \{3, 5, 7, 9, 11\}, B = \{2, 4, 6, \ldots, 100\}, C = \{1, 3, 5, 7, \ldots\}$$

$$D = \{1, 2, 3, 2, 1\}, E = \{x \mid x \text{ is odd, and } x < 12\}$$

n(A) = ?
n(B) = ?
n(C) = ?
n(D) = ?
n(E) = ?

Set Equality

Set Equality: the sets A and B are equal (written $A = B$) provided:

1. every element of A is an element of B, and
2. every element of B is an element of A

In other words, if and only if they contain exactly the same elements

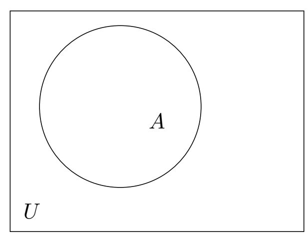
{ a, b, c } = { b, c, a } = { a, b, a, b, c } ?
$\{ 3 \} = \{ x \mid x \in \mathbb{N}$ and $1 < x < 5\}$?
$\{ x \mid x \in \mathbb{N}$ and $x < 0\} = \{ y \mid y \in \mathbb{Q}$ and y is irrational$\}$?

Venn Diagrams & Subsets

Universe of Discourse – the set containing all elements under discussion for a particular problem

In mathematics, this is called the **universal set** and is denoted by U

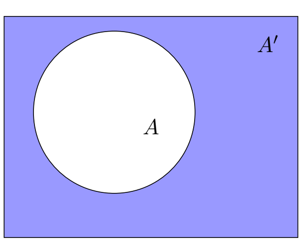
Venn diagrams can be used to represent sets and their relationships to each other



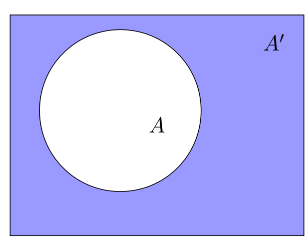
Venn Diagrams

The "Universe" is represented by the rectangle

Sets are represented with circles, shaded regions, and other shapes within the rectangle.



The Complement of a Set



The set A', the shaded region, is the complement of A

A' is the set of all objects in the universe of discourse that are not elements of A

$$A' = \{x \mid x \in U \text{ and } x \notin A\}$$

Let $U = \{1, 2, 3, \ldots, 8\}$, $R = \{1, 2, 5, 6\}$, and $S = \{2, 4, 5, 7, 8\}$

What is: R', the complement of R ?

What is: S', the complement of S ?

What is: U', the complement of U ?

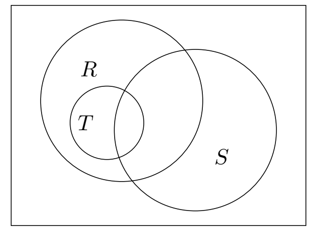
What is: $\emptyset'$, the complement of $\emptyset$?

Subsets

Set A is a **subset** of set B if every element of A is also an element of B. In other words, B contains all of the elements of A.

This is denoted $A \subseteq B$.

Of the sets R, S, and T shown in the Venn diagram below, which are subsets?

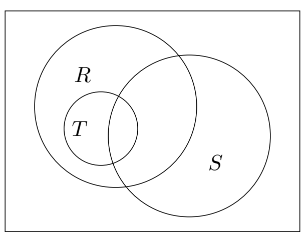


Subsets

Suppose $U = \{1, 2, 3, \ldots, 8\}$, $R = \{1, 2, 5, 6\}$, $S = \{2, 4, 5, 7, 8\}$, and $T = \{2, 6\}$

What element(s) are in the area where all the sets overlap?

What element(s) are in the area outside all the sets?



Set Equality and Proper Subsets

Another definition for **set equality**: Sets A and B are equal if and only if:

1. $A \subseteq B$ and
2. $B \subseteq A$

Proper Subset: $A \subset B$ if $A \subseteq B$ and $A \neq B$

Is or Is Not a Subset?

Is the left set a subset of the set on the right?

$\{a, b, c\}$	$\{a, c, d, f\}$
$\{a, b, c\}$	$\{c, a, b\}$
$\{a, b, c\}$	$\{a, b, c\}$
$\{a\}$	$\{a, b, c\}$
$\{a, c\}$	$\{a, b, c, d\}$
$\{a, c\}$	$\{a, b, d, e, f\}$
X	X
$\emptyset$	$\{a, b, c\}$
$\emptyset$	$\emptyset$

Notes

- Any set is a subset of itself
- Any set is a subset of the universal set
- The empty set is a subset of every set including itself

Set Equality

Is the left set **equal** to, a **proper subset** of, or **not a subset** of the set on the right?

$\{1, 2, 3\}$	$\mathbb{I}$
$\{a, b\}$	$\{a\}$
$\{a\}$	$\{a, b\}$
$\{a, b, c\}$	$\{a, d, e, g\}$
$\{a, b, c\}$	$\{a, a, c, b, c\}$
$\{\emptyset\}$	$\{a, b, c\}$
$\{\emptyset\}$	$\{\}$

Cardinality of the Power Set

Power Set: $\mathscr{P}(A)$ is the set of **all** possible subsets of the set A

For example, if $A = \{0, 1\}$, then $\mathscr{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$

Find the following power sets and determine their cardinality.

$\mathscr{P}(\emptyset) =$

$\mathscr{P}(\{a\}) =$

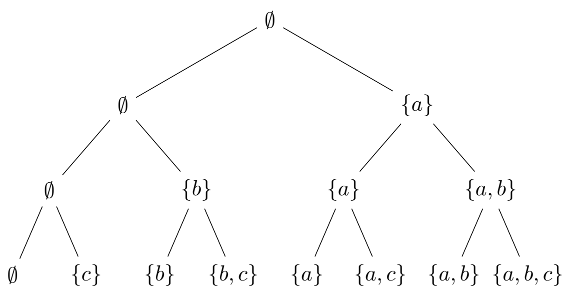
$\mathscr{P}(\{a, b\}) =$

$\mathscr{P}(\{a, b, c\}) =$

Is there a pattern?

Another Method for Generating Power Sets

A **tree diagram** can be used to generate $\mathscr{P}(A)$. Each element of the set is either in a particular subset, or it's not.



The number of subsets of a set with cardinality n is 2^n

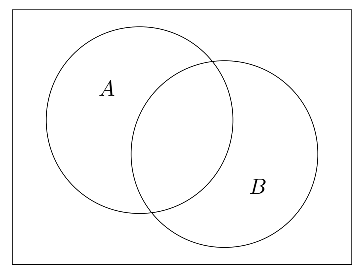
The number of **proper** subsets is $2^n - 1$ (Why?)

Set Operations

Intersection

The **intersection** of two sets, $A \cap B$, is the set of elements common to both: $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

In other words, for an object to be in $A \cap B$ it must be a member of both A **and** B.



Find the Following Intersections

$\{a, b, c\} \cap \{b, f, g\} =$ ________

$\{a, b, c\} \cap \{a, b, c\} =$ ________

For any A, $A \cap A =$ ________

$\{a, b, c\} \cap \{a, b, z\} =$ ________

$\{a, b, c\} \cap \{x, y, z\} =$ ________

$\{a, b, c\} \cap \emptyset =$ ________

For any A, $A \cap \emptyset =$ ________

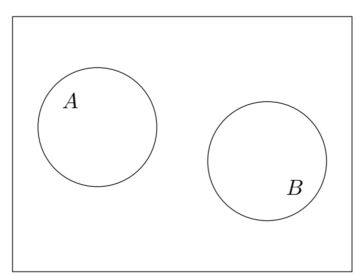
For any A, $A \cap U =$ ________

For any $A \subseteq B$, $A \cap B =$ ________

Disjoint Sets

Disjoint sets: two sets which have no elements in common.

I.e., their intersection is empty: $A \cap B = \emptyset$



Are the Following Sets Disjoint?

{ a, b, c } and { d, e, f, g } ________

{ a, b, c } and { a, b, c } ________

{ a, b, c } and { a, b, z } ________

{ a, b, c } and { x, y, z } ________

{ a, b, c } and $\emptyset$ ________

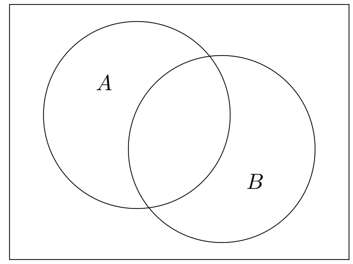
For any A, A and $\emptyset$ ________

For any A, A and A' ________

Set Union

The **union** of two sets, $A \cup B$, is the set of elements belonging to either of the sets: $A \cup B = \{x | x \in A \text{ or } x \in B\}$

In other words, for an object to be in $A \cup B$ it must be a member of either A **or** B.



Find the Following Unions

{ a, b, c } $\cup$ { b, f, g } = ________

{ a, b, c } $\cup$ { a, b, c } = ________

For any A, $A \cup A$ = ________

{ a, b, c } $\cup$ { a, b, z } = ________

{ a, b, c } $\cup$ { x, y, z } = ________

{ a, b, c } $\cup$ $\emptyset$ = ________

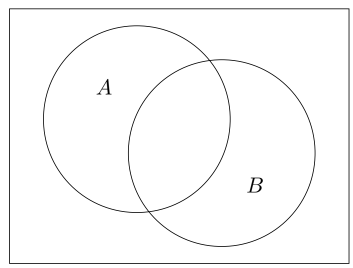
For any A, $A \cup \emptyset$ = ________

For any A, $A \cup U$ = ________

For any $A \subseteq B$, $A \cup B$ = ________

Set Difference

The **difference** of two sets, $A - B$, is the set of elements belonging to set A and **not** to set B: $A - B = \{x | x \in A \text{ and } x \notin B\}$



Note: $x \notin B$ means $x \in B'$

Thus, $A - B = \{x | x \in A \text{ and } x \in B'\}$
$= A \cap B'$

Set Difference Example

$\{1, 2, 3, 4, 5\} - \{2, 4, 6\} =$

$\{2, 4, 6\} - \{1, 2, 3, 4, 5\} =$

Note, in general, $A - B \neq B - A$

Given the sets:

$$U = \{1, 2, 3, 4, 5, 6, 9\}$$
$$A = \{1, 2, 3, 4\}$$
$$B = \{2, 4, 6\}$$
$$C = \{1, 3, 6, 9\}$$

Find each of these sets:

- $A \cup B =$
- $A \cap B =$
- $A \cap U =$
- $A \cup U =$

Given the sets:

$$\begin{aligned} U &= \{1,2,3,4,5,6,9\} \\ A &= \{1,2,3,4\} \\ B &= \{2,4,6\} \\ C &= \{1,3,6,9\} \end{aligned}$$

Find each of these sets:

- $A' =$
- $A' \cap B =$
- $A' \cup B =$
- $A \cup B \cup C =$
- $A \cap B \cap C =$

Describe the Following Sets in Words

- $A' \cup B' =$
- $A' \cap B' =$
- $A \cap (B \cup C) =$
- $(A' \cup C) \cap B =$

Given the sets:

$$\begin{aligned} U &= \{1,2,3,4,5,6,7\} \\ A &= \{1,2,3,4,5,6\} \\ B &= \{2,3,6\} \\ C &= \{3,5,7\} \end{aligned}$$

Find each set:

- $A - B =$
- $B - A =$
- $(A - B) \cup C' =$

Set Operations

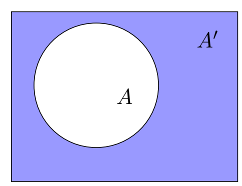
Finding intersections, unions, differences, and complements of sets are examples of **set operations**

An **operation** is a rule or procedure by which one or more objects are used to obtain another object (usually a set or number).

Common Set Operations

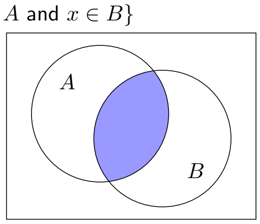
Let A and B be any sets, with U the universal set.

- The **complement** of A is: $A' = \{x | x \in U \text{ and } x \notin A\}$

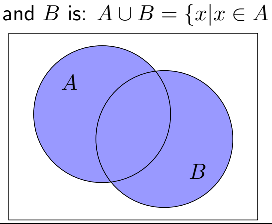


Set Intersection and Union

- The **intersection** of A and B is: $A \cap B = \{x | x \in A \text{ and } x \in B\}$

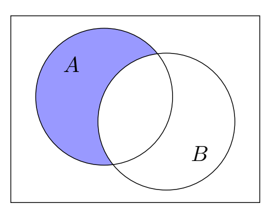


- The **union** of A and B is: $A \cup B = \{x | x \in A \text{ or } x \in B\}$



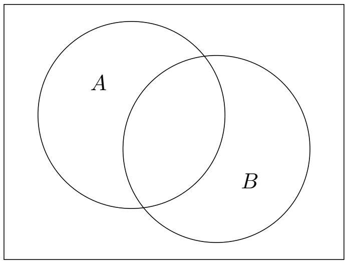
Set Difference

The **difference** of A and B is: $A - B = \{x | x \in A \text{ and } x \notin B\}$

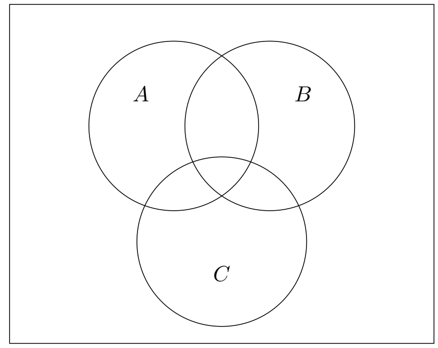


Suppose $U = \{q, r, s, t, u, v, w, x, y, z\}$,
$A = \{r, s, t, i, v\}$,
and $B = \{t, v, x\}$

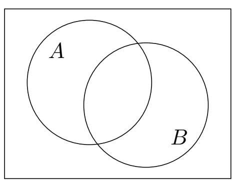
Complete the Venn Diagram to represent U, A, and B



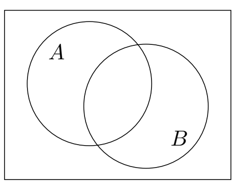
Shade the Diagram for: $A' \cap B' \cap C$



Shade the Diagram for: $(A \cap B)'$



Shade the Diagram for: $A' \cup B'$

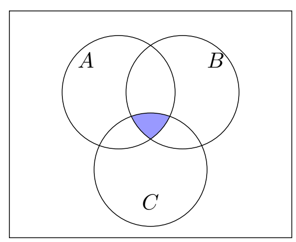


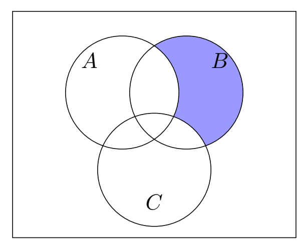
De Morgan's Laws

De Morgan's Laws: For any sets A and B

- $(A \cap B)' = A' \cup B'$
- $(A \cup B)' = A' \cap B'$

Using A, B, C, $\cap$, $\cup$, $-$, and $'$, give a symbolic description of the shaded area in each of the following diagrams. Is there more than one way to describe each?

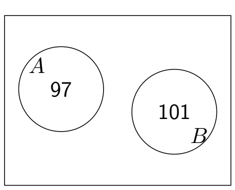




Cardinal Numbers & Surveys

Suppose,
$U =$ The set of all students at EIU
$A =$ The set of all male 2120 students
$B =$ The set of all female 2120 students
$A \cup B =$

Now suppose,
$n(A) = 97$
$n(B) = 101$

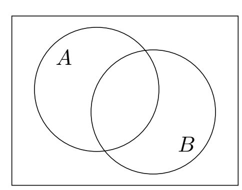


$n(A \cup B) =$

Counting via Venn Diagrams

Suppose,
$U =$ The set of all students at EIU
$A =$ The set of all 2120 students that own a car
$B =$ The set of all 2120 students that own a truck
$A \cup B =$
$A \cap B =$

Now suppose,
$n(A) =$ 33
$n(B) =$ 27
$n(A \cap B) =$ 10
$n(A \cup B) =$



Inclusion/Exclusion Principle

For any two finite sets A and B:
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

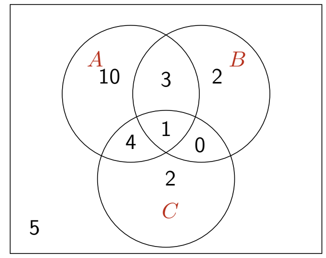
In other words, the number of elements in the union of two sets is the sum of the number of elements in each of the sets minus the number of elements in their intersection.

How many integers between 1 and 100 are divisible by 2 or 5?

Let,

$$A = \{n \mid 1 \leq n \leq 100 \text{ and } n \text{ is divisible by } 2\}$$
$$B = \{n \mid 1 \leq n \leq 100 \text{ and } n \text{ is divisible by } 5\}$$
$$n(A) =$$
$$n(B) =$$
$$n(A \cap B) =$$
$$n(A \cup B) =$$

Back to Counting with Venn Diagrams

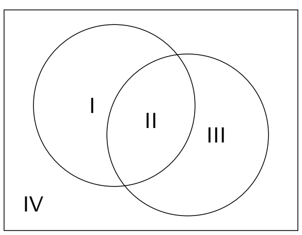


Find the cardinality of the sets:

A	____	$(A \cup B) \cap C$	____
B	____	A'	____
$A \cap B \cap C'$	____	$C - B$	____
$A \cup B$	____	$(A \cup B) \cap C'$	____

Venn Diagram for 2 Sets

There are four disjoint regions



I: $A \cap B'$

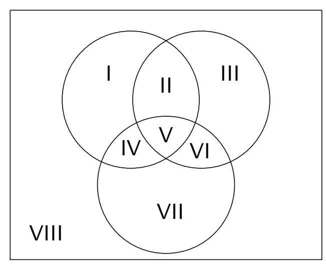
II: $A \cap B$

III: $A' \cap B$

IV: $A' \cap B'$

Venn Diagram for 3 Sets

There are eight disjoint regions



I: $A \cap B' \cap C'$

II: $A \cap B \cap C'$

III: $A' \cap B \cap C'$

IV: $A \cap B' \cap C$

V: $A \cap B \cap C$

VI: $A' \cap B \cap C$

VII: $A' \cap B' \cap C$

VIII: $A' \cap B' \cap C'$

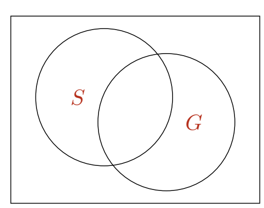
Technique for Counting with Venn Diagrams

- Designate the universal set
- Describe the sets of interest
- Draw a general Venn diagram
- Relate known information to the sizes of the disjoint regions of the diagram
- Infer the sizes of any remaining regions

Using Venn Diagrams to Display Survey Data

Kim is a fan of the music of Paul Simon and Art Garfunkel. In her collection of 22 CDs, she has the following:

- 5 on which both Simon and Garfunkel sing
- 8 total on which Simon sings
- 7 total on which Garfunkel sings
- 12 on which neither Simon nor Garfunkel sings



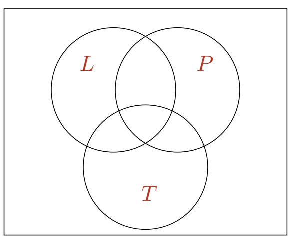
1. How many of her CDs feature only Paul Simon?
2. How many of her CDs feature only Art Garfunkel?
3. How many feature at least one of these two artists?

Love, Prison, and Trucks

There is the cliche that Country–Western songs emphasize three basic themes: love, prison, and trucks. A survey of the local Country–Western radio station produced the following data of songs about:

- 12 truck drivers in love while in prison
- 13 prisoners in love
- 28 people in love
- 18 truck drivers in love
- 3 truck drivers in prison who are not in love
- 2 prisoners not in love and not driving trucks
- 8 people who are out of prison, are not in love, and do not drive trucks
- 16 truck drivers who are not in prison

Number	Love?	Prison?	Trucks?
12			
13			
28			
18			
3			
2			
8			
16			



How many songs were...

1. Surveyed?
2. About truck drivers?
3. About prisoners?
4. About truck drivers in prison?
5. About people not in prison?
6. About people not in love?

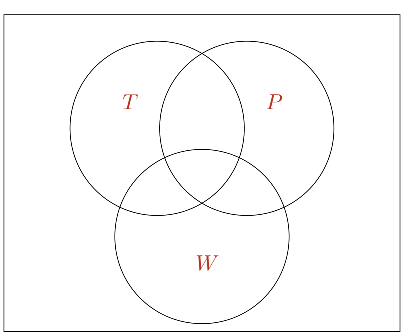
Catching Errors

Jim Donahue was a section chief for an electric utility company. The employees in his section cut down tall trees (T), climbed poles (P), and spliced wire (W). Donahue submitted the following report to the his manager:

$$\begin{aligned} n(T) &= 45 & n(P\cap W) &= 20 \\ n(P) &= 50 & n(T\cap W) &= 25 \\ n(W) &= 57 & n(T\cap P\cap W) &= 11 \\ n(T\cap P) &= 28 & n(T'\cap P'\cap W') &= 9 \end{aligned}$$

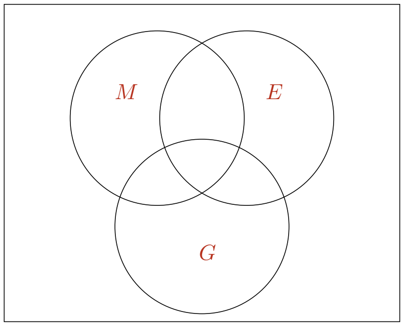
Donahue also stated that 100 employees were included in the report. Why did management reassign him to a new section?

$$\begin{aligned} n(T) &= 45 & n(P\cap W) &= 20 \\ n(P) &= 50 & n(T\cap W) &= 25 \\ n(W) &= 57 & n(T\cap P\cap W) &= 11 \\ n(T\cap P) &= 28 & n(T'\cap P'\cap W') &= 9 \end{aligned}$$



Jim Donahue was reassigned to the home economics department of the electric utility company. he interviewed 140 people in a suburban shopping center to find out some of their cooking habits. He obtained the following results. There is a job opening in Siberia. Should he be reassigned yet again?

- 58 use microwave ovens
- 63 use electric ranges
- 58 use gas ranges
- 19 use microwave ovens and electric ranges
- 17 use microwave ovens and gas ranges
- 4 use both gas and electric ranges
- 1 uses all three
- 2 cook only with solar energy

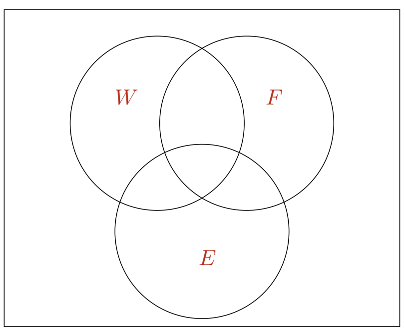


Student Values

Julie Ward, who sells college textbooks, interviewed freshmen on a community college campus to determine what is important to today's students. She found that **Wealth**, **Family**, and **Expertise** topped the list. Her findings can be summarized as:

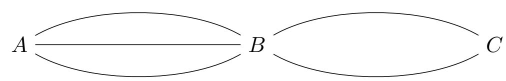
n(W)	=	160	n(E ∩ F)	=	90
n(F)	=	140	n(W ∩ F ∩ E)	=	80
n(E)	=	130	n(E′)	=	95
n(W ∩ F)	=	95	n[(W ∪ F ∪ E)′]	=	10

How many students were interviewed?



How many students were interviewed?

Counting Principles



We have three choices from A to B and two choices from B to C.

How many ways are there to get from A to C through B?

Multiplication Principle

If k operations (events, actions,...) are performed in succession where:

Operation 1 can be done in n_1 ways
Operation 2 can be done in n_2 ways
⋮
Operation k can be done in n_k ways

then the total number of ways the k operations can **all** be performed is:

$$n_1 * n_2 * n_3 * \cdots * n_k$$

In other words, if you have several actions to do and you must do them all you multiply the number of choices to find the total number of choices.

Examples

How many outcomes can there be from three flips of a coin?

Action 1:	Flip a coin
Action 2:	Flip a coin
Action 3:	Flip a coin
Total	

How many ways are there to form a three letter sequence from the letters in $\{A, B, C, \ldots, Z\}$?

Action 1:	Pick a letter
Action 2:	Pick a letter
Action 3:	Pick a letter
Total	

How many ways are there to form a three letter sequence from the letters in $\{A, B, C, \ldots, Z\}$ without repeating any letter?

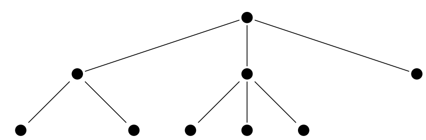
Action 1:	Pick a letter
Action 2:	Pick an unused letter
Action 3:	Pick an unused letter
Total	

How many ways are there to form a license plate that starts with three uppercase letters and end with 3 digits (0–9)?

Action 1:	Pick 3 letters
Action 2:	Pick 3 digits
Total	

Counting with Trees

Tree diagrams consist of nodes (the circles) and branches that connect some nodes.



The nodes represent the possible "states" of a situation.

Branches are the ways or "choices" we have to move to another state.

The "top" node is called the root and it represents the starting state.

Leaves, nodes with no other nodes under them, represent an ending state.

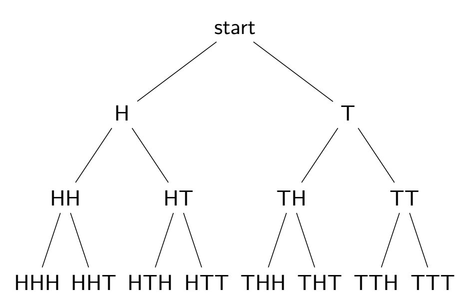
Counting with Trees

This leads to the following technique:

- Use a tree diagram to illustrate a situation
- Count the number of leaves to find the number of possible outcomes

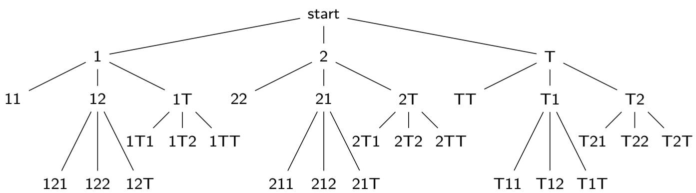
Examples

How many outcomes can there be from three flips of a coin?



8 outcomes

How many ways can a best of three Chess series end. Must have a majority to be declared victor.



21 outcomes

Addition Principle

If $A \cap B = \emptyset$, then $n(A \cup B) =$.

If we are to perform **one** of k operations (events, actions,...) where:

Operation 1 can be done in n_1 ways
Operation 2 can be done in n_2 ways
$\vdots$
Operation k can be done in n_k ways

then the total number of choices is:

$$n_1 + n_2 + n_3 + \cdots + n_k$$

In other words, if you have several actions to do and you are only going to do one of them you add the number of choices to find the total number of choices.

Examples

You are hungry and want to order a combo meal from either Taco Hut or Burger Lord. Taco Hut has 6 different combo meals and Burger Lord has 9. How many choices to you have?

Action 1:	Order a combo from Taco Hut
Action 2:	Order a combo from Burger Lord
Total	

You are hungry and want to order a pizza from either Pizza Place or Pizza Hog. Pizza Place has 6 different toppings and Pizza Hog has 9. Topping can either be on or off of a pizza. How many choices to you have?

Action 1:	Order a pizza from Pizza Place
Action 2:	Order a pizza from Pizza Hog
Total	