Module-2

Relations and Function

Module 2

Relations and Functions

2.1.Relations

A relation is used to describe certain properties of things. That way, certain things may be connected in some way; this is called a relation.It is clear, that things are either related, or they are not, there are no inbetween.

2.1.1.Cartesianproduct

Consider two arbitrary sets X and Y. These to fall ordered pairs(x,y) where x∈X and y∈Y is called the **product, or Cartesian product**, of X and Y. A short designation of this product is $X \times Y$, which is read “X cross Y.” By definition, $X \times Y = \{(x, y) | x \in X and y \in Y\}$

Example(2.1) Let$X = \{1,2\}$and$Y = \{10,15,20\}$. Then

$$X \times Y = \{(1,10), (1,15), (1,20), (2,10), (2,15), (2,20)\}$$

$$Y \times X = \{(10,1), (15,1), (20,1), (10,2), (15,2), (20,2)\}$$

$$\text{Also}, X \times X = \{(1,1), (1,2), (2,1), (2,2)\}$$

One frequently writes X^2 instead of $X \times X$.

There are two things worth noting in the above examples.

First of all $X \times Y \neq Y \times X$. The Cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important. Secondly, using n(S) for the number of elements in a set S, we have:

$$n(X \times Y) = n(X) \times n(Y) = 2 \times 3 = 6$$

The use of the term "relation" is often used as shorthand to refer to binary relations.

A **relation** from a set X to a set Y is any subset of the Cartesian product $X \times Y$.

Definition: Let X and Y be sets. Any set off order pairs(x, y) called a relation in x and y. Further more, the first components in the ordered pairs is called the **domain** of the relation and the set of second ordered pairs is called the **range** of the relation.

A relation from X to Y is a subset of $X \times Y$.

Suppose R is a relation from X to Y. Then R is a set of ordered pairs where each first element comes from X and each second element comes from Y. That is, for each pair x∈X and y∈Y, exactly one of the following is true:

i. $(x, y) \in R$; we then say "x is R − related to y", written xRy.

ii. $(x, y) \notin R$; we then say "x is not R − related to y", written xRy /

If R is a relation from a set X to itself, thatis, if R is a subset of $X^2 = X \times X$,then R is a relation on X.

Example(2.2) Find the domain and range of the relation linking the length of a woman's femur to her height {(45.5, 65.5), (48.2, 68.0), (41.8, 62.2), (46.0, 66.0), (50.4, 70.0)}.

Solution:

Domain: {45.5, 48.2, 41.8, 46.0, 50.4} Set of first coordinates

Range: {65.5, 68.0, 62.2, 66.0, 70.0} Set of second coordinates

Example (2.3)let $A = \{2,3,4\}$and $B = \{3,4,5,6,7\}$. Define the relation R by aRb if and only if a divides b. Find R, Domain of R, Range of R .

Solution

$R = \{(2,4), (2,6), (3,3), (3,6), (4,4)\}$

$\text{Domain} = \{2,3,4\}$

$\text{Range} = \{3,4,6\}$

2.1.2. Representations of Relations

A relation may consist of a finite number of ordered pairs or an infinite number of ordered pairs. Furthermore, a relation may be defined by several different methods:

Arrow diagrams. Venn diagrams and arrows can be used for representing relations between given sets. A relation may be defined by a correspondence (Figure 2.1).The corresponding ordered pairs are {(1, 2), (1, -4), (-3, 4), (3, 4)}.

In the diagram an arrow from x to y means that x is related to y. This kind of graph is called directed graph or digraph.

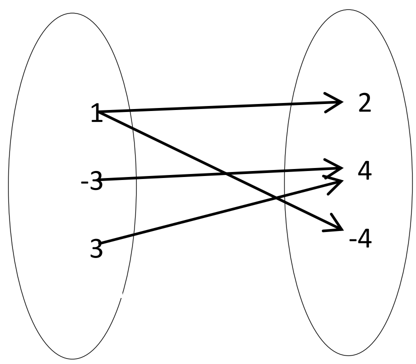


Figure 2.1

Matrix of a Relation. Another way of representing a relation R from A to B is with a matrix. Its rows are labeled with the elements of A, and its columns are labeled with the elements of B. If a ∈ Aand b ∈ B then we write 1 inrow a column b if aRb, otherwise wewrite 0.

For instance the relation R = {(a, 1), (b, 1), (c, 2), (c, 3)} from A = {a, b, c, d} to B = {1, 2, 3, 4} has the following matrix:

1 2 3 4

$$\begin{matrix} a \\ b \\ c \\ d \end{matrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.1.3. Inverse Relation

Given a relation R from A to B, the inverse of R, denoted R^{-1}, is the relation from B to A defined as bR^{-1} a

$$R^{-1} = \{(b, a): (a, b) \in R\}$$

For instance, if R is the relation “being a son or daughter of”, then R^{-1} is the relation “being a parent of”.

Example (2.4) let R = {(1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2)} then

R^{-1} = {(0, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 3)}

Example (2.5) Let R and S be a relations between A and B.

i.e. Show that, if R $\subseteq$ Sthen $R^{-1} \subseteq S^{-1}$.

ii. Prove that $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

Proof (i)

Let $(a, b) \in R^{-1} \rightarrow (b, a) \in R$ definition of inverse relation

$\therefore (b, a) \in S$ since $R \subseteq S$

$\therefore (a, b) \in S^{-1}$ definition of inverse relation

$\therefore R^{-1} \subseteq S^{-1}$ definition of subset

Proof (ii)

(1) Let $(a, b) \in (R \cap S)^{-1}$

$\therefore (b, a) \in (R \cap S)$definition of inverse

$(b, a) \in R$ and $(b, a) \in S$ definition of intersection

$(a, b) \in R^{-1}$and $(b, a) \in S^{-1}$ definition of inverse

$(a, b) \in R^{-1} \cap S^{-1}$ definition of intersection

$(R \cap S)^{-1} \subseteq R^{-1} \cap S^{-1}$ definition of subset

(2) Let $(a, b) \in R^{-1} \cap S^{-1}$

$(a, b) \in R^{-1}$and $(b, a) \in S^{-1}$ definition of intersection

$(b, a) \in R$ and $(b, a) \in S$ definition of inverse

$\therefore (b, a) \in (R \cap S)$definition of intersection

$\therefore (a, b) \in (R \cap S)^{-1}$definition of inverse

$R^{-1} \cap S^{-1} \subseteq (R \cap S)^{-1}$ definition of subset

$\therefore$ from (1)and (2)we have

$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

2.1.4. Composition of Relations

Let A, B and C be three sets.

Given a relation R from A to B and a relation S from B to C, then the composition R_oS of relations R and S is a relation from A to C defined by:

$$R_oS = \{(a,c): (a,b) \in R \text{ and } (b,c) \in S \text{ for some } b \in B\}.$$

For instance, if R is the relation "to be the father of", and S is the relation "to be married to", then R_OS is the relation "to be the father in law of".

Example(2.6) Let $R = \{(1,2),(1,6),(2,4),(3,4),(3,6),(3,8)\}$

$S = \{(2,u),(4,s),(4,t),(6,t),(8,u)\}$ .Find R_oS.

Solution.

$$R_oS = \{(1,u),(1,t),(2,s),(2,t),(3,s),(3,t),(3,u)\}$$

Example(2.7) Let $A = \{0,1,2,3,4,5,6,7\}$,suppose R and T are Two relations on A such that :$R = \{(x,y): 2x + 3y = 15\}, T = \{(x,y): 3x + 2y \in A\}$

Write down R,T and R_oT as a set of ordered pairs ?

Solution

$$R = \{(0,5),(3,3),(6,1)\}$$

$$T = \{(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0)\}$$

$$R_oT = \{(6,0),(6,1),(6,2)\}$$

2.1.5.Properties of Binary Relations

A binary relation R on a set A is called:

1. **Reflexive** if for all $x \in A$, xRx. For instance on Z the relation"equal to" (=) is reflexive.

Example (2.8) Let A = {a, b, c, d} and R be defined as follows:

R = {(a, a), (a, c), (b, a), (b, b), (c, c), (d, c), (d, d)}.

R is a reflexive relation.

Example(2.9) Let R be a relation on a set then if R is reflexive then R^{-1} isreflexive

Proof

Let $(a,a) \in R\ \forall\ a \in A$

$$\therefore (a,a) \in R^{-1}\ \ \forall\ a \in A$$

$\therefore R^{-1}$ is reflexive

2. Transitive if for all $x, y, z \in A$, xRy and y Rz implies xRz.For instance equality (=) and inequality (<) on Z are transitiverelations.

Example (2.10) Let$A = \{a, b, c, d\}$ and R be defined as follows:

$R = \{(a, b), (a, c), (b, d), (a, d), (b, c), (d, c)\}$. Here R is transitive relation on A.

3. Symmetric if for all x, y ∈ A, xRy implies y Rx. For instance on Z, equality (=) is symmetric, but strict inequality (<) is not.

Example(2.11)i. Let $A = \{a, b, c, d\}$and$R = \{(a, a), (b, c), (c, b), (d, d)\}$.

Show that R is symmetric.

ii. Let **R**be these to real numbers and R be the relation aRb if and only if a<b. Show that R is not symmetric.

Solution.

i. bRc and cRb so R is symmetric.

ii. $2 < 4\ but\ 4 \nless 2$.

Example(2.12) let R be a relation on a set A then R is symmetric iff $R = R^{-1}$

Proof

(1) Assume R is a symmetric

$$\text{let } (a, b) \in R \to (b, a) \in R$$

$(b, a) \in R^{-1}$ and $(a, b) \in R^{-1}$

$\therefore R = R^{-1}$

(2) Assume $R = R^{-1}$

let $(a, b) \in R \to (b, a) \in R^{-1}$

$$(b, a) \in R \text{ since } R = R^{-1}$$

$\therefore$ R is symmetric

4. Anti symmetric if for all x, y ∈ A, xRy and y Rx implies x = y.For instance, non-strict inequality (≤) on Z is anti symmetric.

Example(2.13) Let $A = \{a, b, c, d\}$ and R be defined as:

$R = \{(a, b), (b, a), (a, c), (c, d), (d, b)\}$. R is not symmetric, as a R c but $c \not{R} a$.R is not anti-symmetric, because a R b and b R a, but a ≠b.

5. R is irreflexive if, for every a ∈ A , $(a, a) \notin R$

Example (2.14) Let A = {a, b, c, d} and R be defined as follows:

R = {(a, a), (a, c), (b, a), (b, d), (c, c), (d, c), (d, d)}.

Here R is neither reflexive nor irreflexive relation as b is not related to itself and a, c, d are related to themselves.

Example(2.15) let R be a relation on a set A then R is reflexive iff R^c is irreflexive

Proof

Let $(a, a) \in R\ \forall\ a \in A$

$\therefore\ (a, a) \notin R^c\ \ \forall\ a \in A$ definition of complement

$\therefore\ R^c$ is irreflexive definition of irreflexive.

2.1.6. Partial Orders

Definition Let R be a binary relation on a nonempty set X. R is a partial ordering if R is a reflexive, transitive and anti symmetric relation.

For example The relation < is not a partial ordering, since it is transitive and anti symmetric but is not reflexive.

Example (2.16) Let A = {1, 2, 3, 4, 6, 9} and relation R defined on A be "a divides b". Is R Partial ordering relation on A?

Solution

First we list all ordered pairs of R as follows:

R={(1,1),(1,2),(1,3),(1,4),(1,6),(1,9),(2,2),(2,4),2,6),(3,3),(3,6),(3,9),(4,4),(6,6),(9,9)}

(1,1),(2,2),(3,3),4,4),(6,6) and (9,9) ∈ R

$$\therefore\ \textbf{R is reflexive}$$

(1,2),(1,3),(1,4),(1,6),(1,9),(2,4),(2,6),(3,6) and (3,9) ∈ R

but (2,1),(3,1),(4,1),(6,1),(9,1),(4,2),(6,2),(6,3) and (9,3) ∉ R

$\therefore$ **R is antisymmetric**

$(1,2),(2,4) \in R$ and $(1,4) \in R$

$(1,3),(3,6) \in R$ and $(1,6) \in R$

$(1,3),(3,9) \in R$ and $(1,9) \in R$

$\therefore$ **R is transitive**

So R is partial ordering relation

2.1.7. Equivalence Relations

An equivalence relation on a set A is a binary relation on A with the following properties:

1. Reflexive: for all $x \in A$, x R x.
2. Symmetric: x Ry implies yR x.
3. Transitive: (x R y) ^ (y R z) implies xR z.

For instance, on **Z**, the equality (=) is an equivalence relation.

Another example, also on **Z**, is the following: x Ry (mod 2) ("x is congruent to y modulo 2") iff $x - y$ is even. For instance, $6 - 2$ (mod 2) because $6 - 2 = 4$ is even, but $7 \neq 4$ (mod 2), because 7−4 = 3 is noteven. Congruence modulo 2 is in fact an equivalence relation:

1. Reflexive: for every integer x, $x - x = 0$ is indeed even, so $x \equiv x$(mod 2).
2. Symmetric: if $x \equiv y$ (mod 2) then $x - y = t$ is even, but$y - x = -t$ is also even, hence $y \equiv x$ (mod 2).
3. Transitive: assume $x \equiv y \pmod 2$ and $y \equiv z \pmod 2$. Then$x - y = t$and $y - z = u$ are even. From here, $x - z = (x - y) + (y - z) = t + u$ is also even, hence $x \equiv z \pmod 2$.

2.1.8. Equivalence Classes, Quotient Set and Partitions

Given an equivalence relation R on a set A, and an element $x \in A$, the set of elements of A related to x are called the equivalence class of x, represented $[x] = \{y \in A \mid y\,Rx\}$. The collection of equivalence classes, represented $A/R = \{[x] \mid x \in A\}$, is called quotient set of A by R .

One of the main properties of an equivalence relation on a set A_is that the quotient set, i.e. the collection of equivalence classes, isa partition of A. Recall that a partition of a set A is a collection ofnon-empty subsets $A_1, A_2, A_3, \ldots$ of A which are pairwise disjoint and whose union equals A:

1. $A_i \cap A_j = \Phi \quad$ for $i \neq j$

2. $\cup_n A_n = A$

Theorem (2.1) Let R be an equivalence relation on a set A. Then A|R is a partition of A. Specifically:

(i) For each a in A, we have $a \in [a]$.

(ii) [a] = [b] if and only if (a, b) ∈R.

(iii) If $[a] \neq [b]$, then [a] and [b] are disjoint.

Conversely, given a partition $\{A_i\}$of the set A, there is an equivalence relation *R* on A such that the sets A_i are the equivalence classes.

proof

(i) Since R is reflexive, $(a, a) \in R$ for every $a \in A$ and therefore $a \in [a]$.

(ii) Suppose $(a, b) \in R$. We want to show that $[a] = [b]$.

Let $x \in [b]$; then $(b, x) \in R$. But by hypothesis

$(a, a) \in R$ and so, by transitivity, $(a, x) \in R$. Accordingly $x \in [a]$.

Thus $[b] \subseteq [a]$. To prove that $[a] \subseteq [b]$ we

observe that (a, b) ∈R implies, by symmetry, that (b, a) ∈R .

Then, by a similar argument, we obtain [a]⊆ [b].Consequently, [a] = [b].

On the other hand, if $[a] = [b]$, then, by (i), $b \in [b] = [a]$; hence $(a, b) \in R$.

(iii):We prove the equivalent contrapositive statement:

If $[a] \cap [b] \neq \Phi$then $[a] = [b]$

If $[a] \cap [b] \neq \Phi$, then there exists an element $x \in A$ with $x \in [a] \cap [b]$.

Hence $(a, x) \in R$ and $(b, x) \in R$. By symmetry, $(x, b) \in R$

and by transitivity, $(a, b) \in R$. Consequently by (ii), $[a] = [b]$.

Example (2.17) let A = {1, 2, …., 8}. Let R be the equivalence relation defined by $x \equiv y \bmod (4)$

i. Write R as a set of ordered pairs

ii. Find the partition of A induced by R.

Solution

i. $R = \{(1,1), 1,5), (2,2), (2,6), (3,3), (3,7), (4,4), (4,8), (5,1), (5,5), (6,2), (6,6), (7,3), (7,7), (8,4), (8,8)\}$

ii. [1] = {1, 5}

We pick an element which does not belong to [1], say 2. Those elements related to 2 are 2, and 6, hence

[2] = {2,6}

$[3] = \{3,7\}$

The only element which does not belong to [1] , [2] or [3] is 4. The only element related to 4 is 4. Thus

[4] = {4,8}

Accordingly, the following is the partition of A induced by R:

$A|R = \{[1], [2], [3], [4]\}$

2.2.Functions

In this section we introduce a special type of relation called a **function.**

Definition: Given a relation in x and y, we say "y is a **function** of x" if for every element x in the domain, there corresponds exactly one element y in the range.

Note that the definition of a function requires that a relation must be satisfying two conditions in order to qualify as a function:

The first condition is that every $x \in X$ must be related to $y \in Y$ that is the domain of f must be X and not merely a subset of X

The second requirement of uniqueness can be expressed as:

$$(x, y) \in f\text{and } (x, z) \in f \implies y = z$$

Sometimes we represent the function with a diagram like this: $f: A \rightarrow B$ or $A \xrightarrow{f} B$

For instance, the following represents the function from Z to Z defined by $f(x) = 2x + 1.$

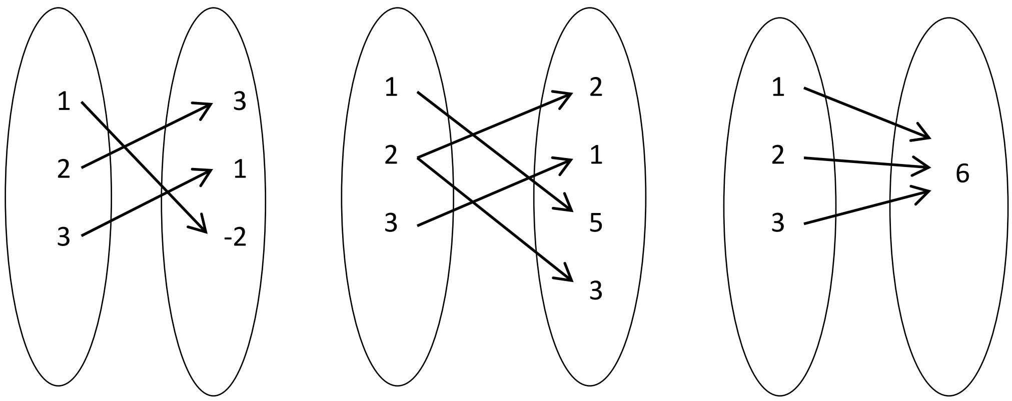
The element y = f(x) is called the image of x, and x is a pre image of y.

Remark: Functions are sometimes also called **mappings** or **transformations**.

To understand the difference between a relation that is a function and a relation that is not a function.

Example (2.18) Determine which of the relations define y as a function of x.

a. **b.** **c.**



Solution

a. This relation is defined by the set of ordered pairs$\{(1,-2),(2,3),(3,1)\}$

Notice that for each x in the domain there is only one corresponding y in the range. Therefore, this relation is a function.

When $x = 1$, there is only one possibility for y: $y = -2$

When $x = 2$, there is only one possibility for y: $y = 3$

When $x = 3$, there is only one possibility for y: $y = 1$

b. This relation is defined by the set of ordered pairs

Same x

$\{(1,5),(2,2),(2,3),(3,1)\}$

Different y

When $x = 1$,there are two possible range elements: $y = 2$ and $y = 3$

Therefore, this relation is not a function.

c. This relation is defined by the set of ordered pairs$\{(1,6),(2,6),(3,6)\}$

when $x = 1$, there is only one possibility for y: $y = 6$

when $x = 2$, there is only one possibility for y: $y = 6$

when $x = 3$, there is only one possibility for y: $y = 6$

Because each value of x in the domain has only one corresponding y value, this relation is a function.

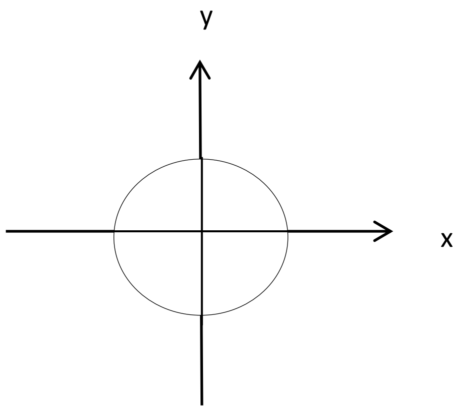
Remark: Vertical Line Test

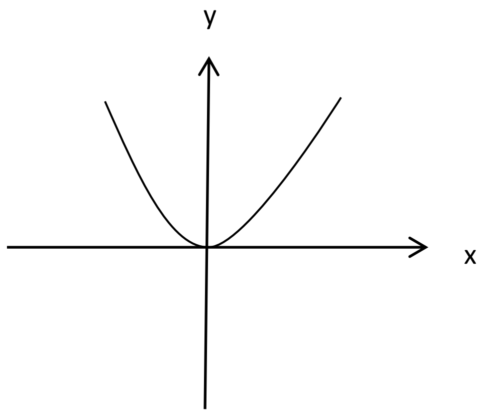
A relation that is not a function has at least one domain element x paired with more than one range value y. For example, the ordered pairs (4, 2) and (4, -2) do not constitute a function because two different y-values correspond to the same x. These two points are aligned vertically in the xy-plane, and a vertical line drawn through one point also intersects the other point. Thus, if a vertical line drawn through a graph of a relation intersects the graph in more than one point, the relation cannot be a function.

This idea is stated formally as the **vertical line test**.

Example(2.19) Use the vertical line test to determine whether the following relations define y as a function of x.

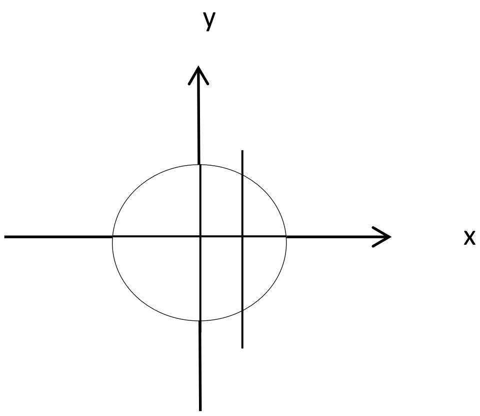
a. **b.**





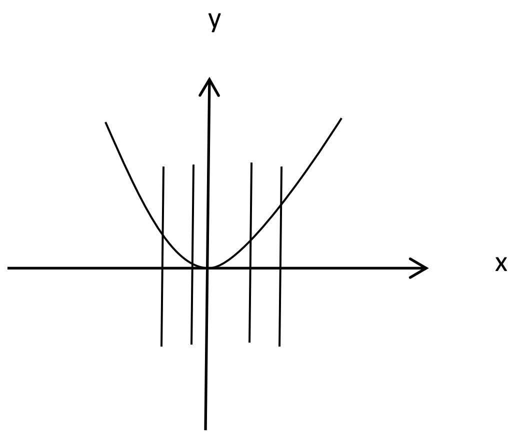
Solution

a. **b.**



Not function

A vertical line intersects
in more than one point.



Function

No vertical line intersects
in more than on point

A function is defined as a relation with the added restriction that each value in the domain must have only one corresponding y-value in the range. In mathematics, functions are often given by rules or equations to define the relationship between two or more variables. For example, the equationy $= 3x$

defines the set of ordered pairs such that the y-value is 3times the x-value.

When a function is defined by an equation, we often use **function notation**.

For example, the equation $y = 3x$ can be written in function notation as $f(x) = 3x$ Where f is the name of the function, x is an input value from the domain of the function, and f (x) is the function value (or y-value) corresponding to x

The notation f (x) is read as "f of x" or "the value of the function f at x."

A function may be evaluated at different values of x by substituting x-values from the domain into the function. For example, to evaluate the function defined by $f(x) = 3x$ at $x = 4$, substitute $x = 4$ in to the function

$$\begin{array}{c} f(x) = 3\,x \\ \downarrow \quad \downarrow \\ f(4) = 3(4) \\ f(4) = 12 \end{array}$$

The function value $f(4) = 12$ can be written as the ordered pair (4, 12**)**thus, when $x = 4$ the corresponding function value is 12.We say "f of 4 is 12"or "f at 4 is 12." The names of functions are often given by either lowercase or upper case letters, such as f, g, h, p, K, and M.

Example(2.20) Given the function defined by $f(x) = 2x - 1$.

find the function values i. $f(0)$ ii. $f(1)$ iii. $f(-1)$ iv. $f(2)$

Solution

i. $f(0) = 2(0) - 1$

$= -1$ We say, "f of 0 is -1." This is equivalent to the ordered pair $(0, -1)$

ii. $f(1) = 2(1) - 1$

$= 1$ We say, "f of 1 is 1." This is equivalent to the ordered pair $(1, 1)$

iii. $f(-1) = 2(-1) - 1$

$= -3$ We say, "f of -1 is -3." This is equivalent to the ordered pair $(-1, -3)$

iv. $f(2) = 2(2) - 1$

$= 4 - 1$

$= 3$ We say, "f of 2 is 3." This is equivalent to the ordered pair $(2,3)$

2.2.1. Domain and Range of a Function

A function is a relation, and it is often necessary to determine its domain andrange. Consider a function defined by the equation $\mathbf{y} = \mathbf{f}(\mathbf{x})$The**domain** of *f* isthe set of all *x*-values that when substituted into the function, produce a real number.

The **range** of *f* is the set of all *y*-values corresponding to the values of *x* in the domain.

To find the domain of a function defined by$\mathbf{y} = \mathbf{f}(\mathbf{x})$, keep these guidelines in mind.

- Exclude values of *x* that make the denominator of a fraction zero.
- Exclude values of *x* that make a negative value within a square root.

Example(2.21) Find the domain of the functions. Write the answers in interval notation..

$$a.\, f(x) = \frac{x+7}{2x-1} \qquad b.\, g(x) = \frac{x-4}{x^2+9}$$

Solution

a. The function will be undefined when the denominator is zero, that is, when

$2x - 1 = 0$

$2x = 1$

$x = \frac{1}{2}$The value $x = \frac{1}{2}$must be excluded from the domain.

Interval notation:$\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

b. The quantity is greater than or equal to 0 for all real numbers x, and the number 9 is positive. Therefore, the sum must be positive for all real numbers x. The denominator of $g(x) = \frac{x-4}{x^2+9}$ will never be zero;the domain is the set of all realnumbers. Interval notation: $(-\infty, +\infty)$

2.2.2. Graphs of Basic Functions

We can associate a set of pairs in A× B to each function from A to B. This set of pairs is called the **graph** of the function and is often displayed pictorially to aid in understanding the behavior of the function

Definition Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs {(a, b) | a ∈A and f (a) = b}.

From the definition, the graph of a function f from A to B is the subset of A× B containing the ordered pairs with the second entry equal to the element of B assigned by f to the first entry.

Also, note that the graph of a function f from A to B is the same as the relation from A to B determined by the function f .

To determine the shapes of the basic functions, we can plot several points to establish the pattern of the graph. Analyzing the equation itself may also provide insight to the domain, range, and shape of the function.

Example(2.22) Graph the functions defined by $f(x) = x^2$

Solution

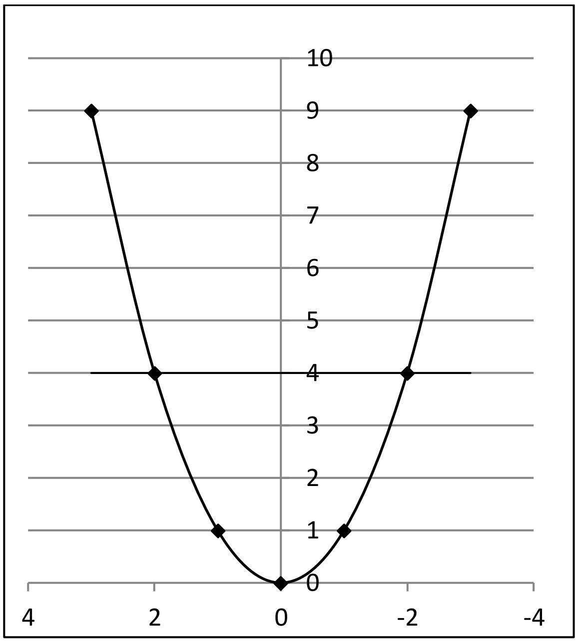
The domain of the function given by $f(x) = x^2$or equivalently $y = x^2$ is all real numbers.

To graph the function, choose arbitrary values of *x* with in the domain of the function. Be sure to choose values of *x* that are positive and values that are negative to determine the behavior of the function to the right and left of the origin.

The function values are equated to the square of *x*, s of (x)will always be greater than or equal to zero. Hence, the *y*-coordinates on the graph will never be negative. The range of the function is{y|yis a real number and $y \geq 0$}.

The arrows on each branch of the graph imply that the pattern continues indefinitely

x	$f(x) = x^2$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



Table(2.1)Figure(2.1)

2.2.3. Types of Functions

1. One-to-One or **Injective**: A function$f: A \to B$ is called oneto-one or injective if each element of B is the image of at most one element of A (figure 2.2):

$$\forall x, x' \in A, f(x) = f(x') \Longrightarrow x = x'$$

For instance, $f(x) = 2x$from Z to Z is injective

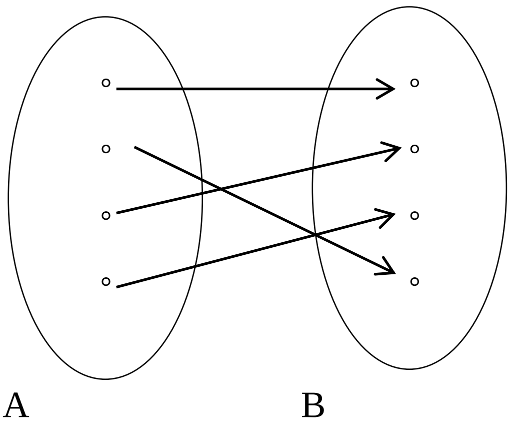


Figure (2.2) One-to-one function

2.Onto or **Surjective**: A function **f**:A→B is called onto or surjective if every element of B is the image of some element of A(fig.2.3):

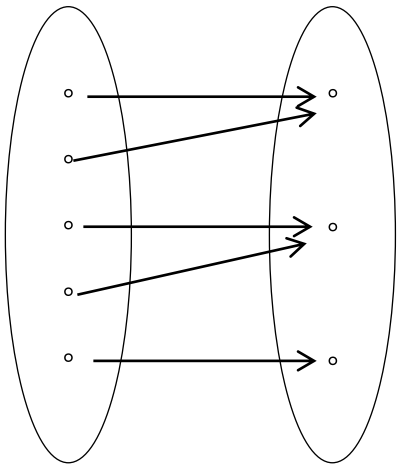


Figure (2.3) Onto function

Example(2.23) Using two-element sets or three-element sets as domains and ranges, find an example of an onto function that is not one-to-one.

Solution

Notice that the function given by $f(1) = a, f(2) = b, f(3) = a$is an example of afunction from $\{1, 2, 3\}$ to $\{a, b\}$that is onto but not one to one.

Example(2.24) Using two-element sets or three-element sets as domains and ranges, find an example of a one-to-one function that is not onto.

Solution

Notice that the function given by $f(1) = c, f(2) = a$is an example of a function from $\{1, 2\}$to $\{a, b, c\}$that is one-to one but not onto.

3. One-To-One Correspondence or **Bijective**: A function $f: A \to B$ is said to be a one-to-one correspondence, or bijective, or abijection, if it is one-to-one and onto (figure 2.4).

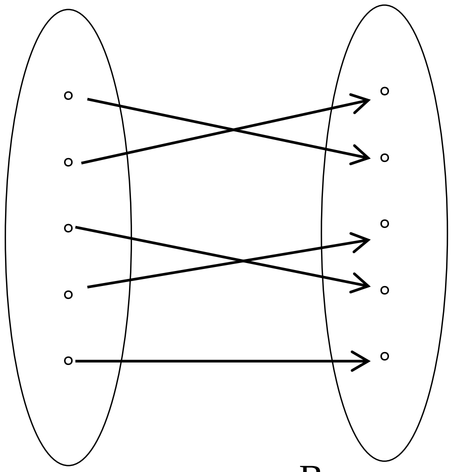


Figure (2.4) Bijection.

2.2.4. Inverse Function. If $f: A \to B$ is a bijective function, its inverse is the function $f^{-1}: B \to A$ such that $f^{-1}(y) = x$ if and only if $f(x) = y$

A characteristic property of the inverse function is that $f^{-1}{}_o f = 1_A$ and $f_o f^{-1} = 1_B$

Example(2.25) let f be the function from {a, b, c} to {1, 2, 3} such that f (a) = 2, f (b) = 3, and f (c) = 1. Is f invertible, and if it is, what is its inverse?

Solution

The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f , so $f^{-1}(1) = c, f^{-1}(2) = a$ and $f^{-1}(3) = b$

Example (2.26) let $f: \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution

The function f has an inverse because it is a one-to-one correspondence, as follows

To reverse the correspondence, suppose that y is the image of x, so that

$y = x + 1$. Then,

$x = y - 1$. This means that $y - 1$ is the unique element of **Z** that is sent to y by f . Consequently $f^{-1}(y) = y - 1$

Definition Identity function. Given a set A, the function $:1_A: A \to B$ defined by $1_A(x) = x$ for every x in A is called the identity function for A.

Remark Geometrical Characterization of one-to-one and onto functions

Consider now functions of the form f :**R** → **R**. Since the graphs of such functions may be plottedin the Cartesian plane $\mathbf{R^2}$ and since functions may be identified with their graphs, we might wonder

whether the concepts of being one-to-one and onto have some geometrical meaning. The answer is yes. Specifically:

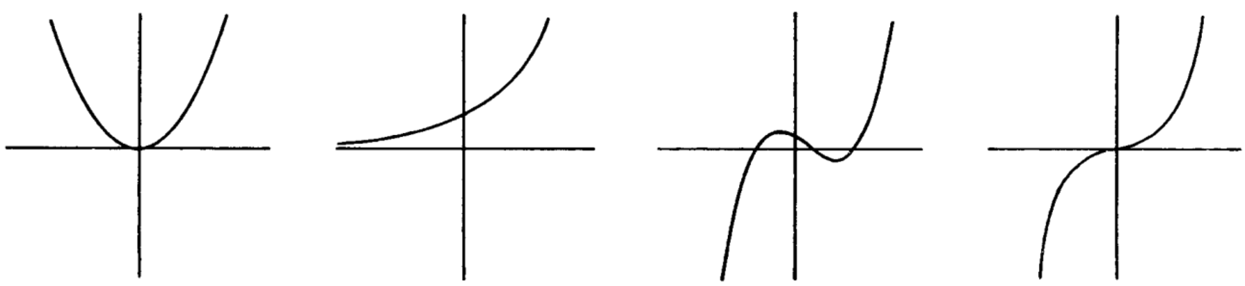
(1) f:**R** → **R** is one-to-one if each horizontal line intersects the graph off in at most one point.

(2) f:**R** → **R** is an onto function if each horizontal line intersects the graph of f at one or more points.

Accordingly, if f is both one-to-one and onto, i.e. invertible, then each horizontal line will intersect the graph off at exactly one point.

Example(2.27) Consider the following four functions from **R** into **R**:

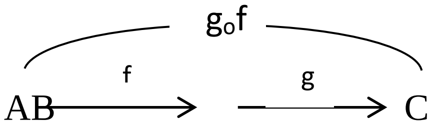
The graphs of these functions appear in Figure2.5. Observe that there are horizontal lines which intersect the graph of f twice and there are horizontal lines which do not intersect the graph of f at all; hence f is neither one to-one nor onto. Similarly, g is one-to-one but not onto, h is onto but not one-to-one and t is both one-to-one and onto. The inverse of t is the cube root function**, i.e.**,$t^{-1}(x) = \sqrt[3]{x}$



$f(x) = x^2$ $\quad$ $g(x) = 2^x$ $\quad$ $h(x) = x^3 - 2x^2 - 5x + 6$ $\quad$ $t(x) = x^3$

Figure (2.5)

2.2.5. Function Composition. Given two functions $f: A \to B$ and $g: B \to C$ the composite function off and g is the function $g_o f = A \to C$ defined by $(g_o f)(x) = g(f(x))$ for every x in A:



In other words, $g_o f$ is the function that assigns to the element a of A the element assigned by g to f(a). That is, to find $(g_o f)(a)$ we first apply the function f to a to obtain f(a) and then we apply the function g to the result f(a) to obtain (gof)(a) = g(f(a)). Note that the composition $g_o f$ cannot be defined unless the range of f is a subset of the domain of g .

Example(2.28) Let g be the function from the set {a, b, c} to itself such that g(a) = b, g(b) = c, and g(c) = a. Let f be the function from the set {a, b, c} to the set {1, 2, 3} such that f (a) = 3, f (b) = 2, and f (c) = 1. What is the composition of f and g, and what is the composition of g and f ?

Solution:

The composition $f_o g$ is defined by $(f_o g)(a)$ = f (g(a)) = f (b) = 2,

$(f_o g)(b) = f(g(b)) = f(c) = 1$, and $(f_o g)(c) = f(g(c)) = f(a) = 3$.

Note that $g_o f$ is not defined, because the range of f is not a subset of the domain of g.

Example(2.29) Let f and g be the functions from the set of integers to the set of integers defined by f (x) = 2x + 3 and g(x) = 3x + 2. What is the composition of f and g? What is the composition of g and f ?

Solution:

Both the compositions $f_o g$ and $g_o f$ are defined. Moreover,

$$(f_o g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g_o f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

Example(2.30) Let functions f and g defined as follows:

$f(x) = 2x + 1$ and $g(x) = x^2 - 2$ respectively, find $g_o f(4)$ and $f_o g(4)$?

Solution:

$$(g_o f)(4) = g(f(4))$$
$$= g(2 \times 4 + 1)$$
$$= g(9)$$
$$= 9^2 - 2$$
$$= 79$$
$$(f_o g)(4) = f(g(4))$$
$$= f(4^2 - 2)$$
$$= f(14)$$
$$= 2 \times 14 + 1$$
$$= 29$$

Note that even though $f_o g$ and $g_o f$ are defined for the functions f and g , $f_o g$ and $g_o f$ are not equal. In other words, the commutative law does not hold for the composition of functions**.**

Some properties of function composition are the following:

1.If $f: A \to B$ is a function from A to B, we have that $f_o 1_A = 1_{B\,o} f = f$

2. Function composition is associative, i.e., given three functions $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$,

we have that $h_o(g_o f) = (h_o g)_o f$

Example(2.31) Let functions f, g and h defined as follows

$f(x) = 3x \qquad f: N \to N$

$g(x) = 2x^2 \qquad g: N \to N$

$h(x) = 5x \qquad h: N \to N$

Where N is Positive integers. Find $(h_o g_o f)(x)$

Solution

$$(h_o g_o f)(x) = (h_o g)_o f(x)$$
$$= (h_o g)(3x)$$

$= h\big(g(3x)\big)$

$= h(2 \times (3x)^2)$

$= h(18x^2)$

$= 5 \times 18x^2$

$= 90x^2$

When the composition of a function and its inverse is formed, in either order, an identity function is obtained. To see this, suppose that f is a one-to-one correspondence from the set A to the set B. Then the inverse function f^{-1}exists and is a one-to-one correspondence from B to A. The inverse function reverses the correspondence of the original function, so $f^{-1}(b) = a$when f (a) = b, and f (a) = b when $f^{-1}(b) = a$. Hence,

$$(f^{-1}{}_o f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a,$$

and

$$(f_o f^{-1})(b) = f(f^{-1}(b)) = f(a) = b,$$

Consequently $f^{-1}{}_o f = I_A$and $f_o f^{-1} = I_B$, where I_A and I_B are the identity functions on the sets A and B, respectively. That is, $(f^{-1})^{-1} = f$.

Problems

2.1. Find the domain and range of the relation.

{(0,0),(-8,4),(2,1),(-3,4),(-8,0)} (Ans: domain={0,-8,2,-3},range={0,4,1})

2.2. Let R be the relation on positive integers defined by:

$R = \{(x, y) \mid x + 3y = 12\}$.

i. Write R as a set of ordered pairs. ii. Find R^{-1}

iii. Find the domain and range of R. iv. Find the composition relation $R_O R$

(Ans: i. {(9, 1), (6, 2), (3, 3)} ii. {(1,9),(2,6),3,3)}

iii. {3, 6, 9},{1,2,3} iv. {(3,3)}

2.3 Let $A = \{1, 2, 3, 4\}$. Define the relation R by aRb if and only if $a \leq b$.

Find, R, Domain of R ,Range of R.

(Ans:{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3),(2,4),(3,3),(3,4),(4,4)},domain = range = A)

2.4. suppose R and S are two relations from A to B then

i. if $R \subseteq S$ then $S^c \subseteq R^c$

ii. $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

iii. $(R \cap S)^c = R^c \cup S^c$

2.5. Let R = {(1, y), (1, z), (3, y)} be a relation from A = {1; 2; 3} to B ={x, y, z}

a. Find R^{-1} ? (Ans:{(y,1),(z,1),(y,3)}

b. Compare $(R^{-1})^{-1}$ and R ? (Ans: $R=(R^{-1})^{-1}$

2.6. Given the following two relations from $A = \{1, 2, 4\}$ to $B = \{2, 6, 8, 10\}$

aRb if and only if $a|b$.

aSb if and only if $b - 4 = a$

List the elements of $R, S, R \cup S$; and $R \cap S$.

(Ans:$R = \{(1, 2), (1, 6), (1, 8), (1, 10), (2, 2), (2, 6), (2, 8), (2, 10); (4, 8)\}$

$S = \{(2, 6); (4, 8)\}$, $R \cup S = R$, $R \cap S = S$

2.7. Let R and S be two relations on a set of positive integers I,

$R = \{(x, 2x)\}: x \in I\}$, $S = \{(x, 7x): x \in I\}$. Find $R_o R$, $R_o R_o R$ and $R_o S_o R$?

(Ans:$R_o R = \{(x, 4x): x \in I\}$, $R_o R_o R = \{(x, 8x): x \in I\}$ and $R_o S_o R\{(x, 28x), x \in I\}$

2.8. Let $W = \{a, b, c, d\}$and F, G, H be realations from w to w such that:

$F = \{(a, b), (b, c), (c, d), (d, a)\}$, $G = \{(a, c), (b, d), (c, a), (d, b)\}$ and

$H = \{(a, d), (b, a), (c, b), (d, c)\}$. Find $(F_o G_o H)^{-1}{}_o H$

(Ans: $(F_o G_o H)^{-1}{}_o H = \{(a, b), (b, c), (c, d), (d, a)\}$

2.9. If R and S are two reflexive relations then $R \cup S$ and $R \cap S$ are refelexive .

2.10. If R is symmetric relation then $R^{-1}$and R^c are symmetric .

2.11. Determine the relation R on the set A = {1,2,3,4,5} whose M_R is given below :

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(Ans: R is partial ordering since it is reflexive, anti symmetric and transitive)

2.12. Let A = {1, 2, ….,16}. Let R be the relation on A defined by “xy is a square,” Find the equivalence class [1].

(Ans: {1,4,9,16})

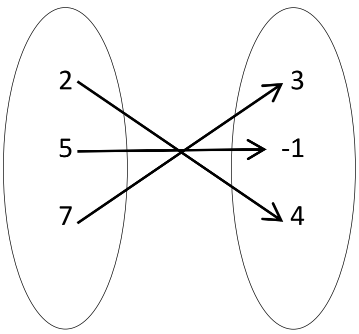
2.13. Let A= {1, 2, . . . , ,20}. Let R be the equivalence relation on S defined by $x \equiv y$ (mod 5), that is, $x - y$ is divisible by 5. Find the partition of A induced by R, i.e. the quotient set A/R.

(Ans: [{1, 6, 11,16}, {2, 7, 12,17}, {3, 8, 13,18}, {4, 9, 14,19},{5, 10, 15,20}])

2.14. Determine if the relations define y as a function of x.

a.$\{(4,2), (-5,4), (0,0), (8,4)\}$. **b.** $\{(-1,6), (8,9), (-1,4), (-3,10)\}$

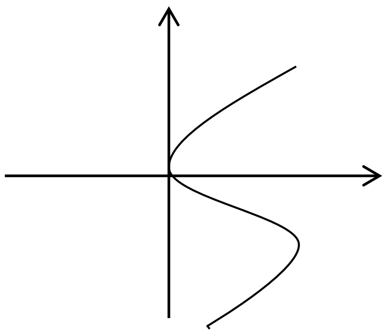
c.



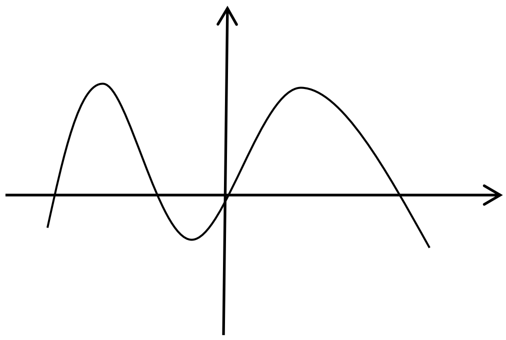
(Ans: a. yes b. No c. Yes)

2.15. Use the vertical line test to determine whether the relations define y as a function of x.

a.



b.



(Ans: a. not a function   b. Function)

2.16. Rewrite each of the following function from **R** into **R**using a formula

a. To each number let g assign its cube.

b. To each number let h assign the number 4.

c. To each nonnegative number let f assign the number 5, and to each positive number let f assign to its square.

$$\left(\textbf{Ans}\text{: a. } g(x) = x^3 \qquad \text{b. } h(x) = 4 \qquad \text{c. } f(x) = \begin{cases} 5 & x \leq 0 \\ x^2 & x > 0 \end{cases}\right)$$

2.17 Given the function defined by$g(x) = \frac{1}{2}x - 3$, find the function values.

$$\text{a. } g(-2) \quad \text{b. } g(1) \quad \text{c. } g(4) \quad \text{d. } g\left(\frac{2}{3}\right)$$

$$\left(\text{Ans: a. } -4 \quad \text{b. } -2.5 \quad \text{c. } -1 \quad \text{d. } \frac{-8}{3}\right)$$

2.18. Given the function defined by,$g(x) = 4x - 3$ find the function values.

$$\text{a. } g(-x) \qquad \text{b. } g(w) \qquad \text{c. } g(x + w)$$

$$(\text{Ans: a. } -4x - 3 \quad \text{b. } 4w - 3 \quad \text{c. } 4x + 4w - 3)$$

2.19. Find the domain of the functions. Write the answers in interval notation.

$$a.\ f(x) = \sqrt{x+4} \qquad b.\ g(x) = x^2 - 3x$$

$$c.\ p(x) = \sqrt{x-2} \qquad d.\ h(x) = \frac{2x+1}{x-9}$$

$\{$Ans: a. $[-4, \infty)$ b. $(-\infty, +\infty)$ c. $(2, \infty)$ d. $(-\infty, 9) \cup (9, \infty)\}$

2.20. Let A be the set of student in a school. Determine which of the following assignment define a function on A.

a. To each student assign his age.

b. To each student assign his friend.

c. To each student assign his gender.

(Ans:a.Yes b.NO c.Yes)

2.21. Let $f(x) = x^2$ Prove that

$$f(x^2 + y^2) = f\big(f(x)\big) + f\big(f(y)\big) + 2f(x).f(y)$$

2.22. Graph the functions defined by a. $g(x) = \frac{1}{x}$ b. $h(x) = |x|$

Hint:a. Domain $(-\infty, 0) \cup (0, \infty$, Range $(-\infty, 0) \cup (0, +\infty)$

b. Domain $(-\infty, +\infty)$, Range $[0, \infty)$

2.23. a. Show that the function $f:$ **R** → **R** defined by $f(x) = 3x - 5$ is surjective.b. Show that the function $f:$ **Z** → **Z** defined by $f(n) = 3n - 5$ is not surjective.

2.24. a. Show that the function $f:$ **R** → **R** defined by $f(x) = 3x - 5$ is bijective b. Show that the function $f:$ **R** → **R** defined by $f(x) = x^2$ is not bijective.

2.25. Consider the following functions:

$f = 3x$ $f: R \to R$

$g = 3^x$ $g: R \to R$

Are f,g invertible functions, why?

(Ans: f invertible function , g is not invertible function)

2.26. Let f: **R** → **R** be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto; hence f has an inverse function f^{-1}. Find a formula for f^{-1}

$\left(\text{Ans: } f^{-1}(x) = \frac{x+3}{2}\right)$

2.27. Let $f(x) = \frac{3}{x-2}$ $f: R - \{2\} \to R - \{0\}$**.** Find $f^{-1}\sqrt{3}$?

(Ans: $f^{-1}(\sqrt{3} = \sqrt{3} + 2)$

2.28. Let $A = \{a, b, c\}, B = \{x, y, z\}, C = \{r, s, t\}$. Let $f: A \to B$ and $g: B \to C$ be defined by: $f = \{(a, y)(b, x), (c, y)\}$ and $g = \{(x, s), (y, t), (z, r)\}$.

Find: (a) composition function $g_o f: A \to C$; (b) $Im(f), Im(g), Im(g_o f)$.

(Ans: $a. g_o f = \{(a, t), (b, s), (c, t)\}$.

$b. Im(f) = \{x, y\}, Im(g) = \{r, s, t\}, Im(g_o f) = \{s, t\}$)

2.29. Let f and g functions defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ respectively find $(f_o g)(a + 2)$ and $(g_o f)(a + 2)$

(Ans: $(f_o g)(a + 2) = 2a^2 + 8a + 5$ and $(g_o f)(a + 2) = 4a^2 + 20a + 23$)

2.30. Let $f: N \to Q^+, f(n) = \frac{n}{n+3}$, $g: Q^+ \to Q$, $g(r) = \frac{1}{r+1}$

If $g(f(d)) = \frac{7}{13}$, what is d ? (Ans: $d = 18$)