

Module-3

Logic and Proofs

Module 3

LOGIC AND PROOFS

3.1 INTRODUCTION

PROPOSITION (OR) STATEMENT:

Proposition is a declarative statement that is either true or false but not both. The truth value of proposition is true or false.

Truth table

It displays the relationship between the truth values of proposition.

Negation of a proposition

If P is a proposition, then its negation is denoted by $\neg P$ or $\sim p$ and is defined by the following truth table.

P	$\neg P$
T	F
F	T

EXAMPLE

P - Ram is intelligent

$\neg P$ - Ram is not intelligent

proposition is a declarative sentence which is either true or false but not both.

COMPOUND PROPOSITION

It is a proposition consisting of two or more simple proposition using logical operators.

3.2 LOGICAL CONNECTIVES

(1) DISJUNCTION (OR)

The disjunction of two proposition P and Q is the proposition $P \vee Q$ [read as P or Q] and is defined by the following truth table.

P	Q	$P \vee Q$
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T	T	T
T	F	T
F	T	T
F	F	F

(1) CONJUNCTION (AND)

If P and Q are two propositions , then the conjunction of P and Q is denoted by $P \wedge Q$ (read as P and Q) and is defined by following truth table.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

CONDITIONAL AND BI- CONDITIONAL PROPOSITION

(1) Conditional proposition

If p and q are propositions, then the implication “If p then q“ denoted by $p \rightarrow q$, called the conditional statement of p and q , is defined by following truth table.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

NOTE

$p \rightarrow q$ is false when p is true and q is false. Otherwise it is true.

The different situations where the conditional statements applied are listed below.

- (1) If p then q
- (2) p only if q
- (3) q whenever p
- (4) q is necessary for p
- (5) q follows from p
- (6) q when p
- (7) p is sufficient for q
- (8) p implies q

Converse, contrapositive and Inverse statement

If $p \rightarrow q$ is a conditional statement, then

- (1) $q \rightarrow p$ is called converse of $p \rightarrow q$
- (2) $\neg q \rightarrow \neg p$ is called contrapositive of $p \rightarrow q$
- (3) $\neg p \rightarrow \neg q$ is called inverse of $p \rightarrow q$

EXAMPLE

p : Ram is a computer science student

q : Ram study DBMS

$p \rightarrow q$: If Ram is a computer science student, then he will study DBMS.

(2) Bi-conditional proposition

If p and q are proposition, then the proposition p if and only if q , denoted by $p \leftrightarrow q$ is called the bi-conditional statement and is defined by the following truth table.

p	q	
T	T	T
T	F	F
F	T	F
F	F	T

NOTE

$P \leftrightarrow Q$ is true if both p and q have same truth values. Otherwise $P \leftrightarrow Q$ is false.

EXAMPLE

P: You can take the flight

q: You buy a ticket

$p \leftrightarrow q$: You can take the flight if and only if buy a ticket.

Symbolize the statements using Logical Connectives

Example: 1

The automated reply can be sent when the file system is full.

P: The automated reply can be sent

Q: The file system is full

Solution:

Symbolic form : $q \rightarrow \neg p$

EXAMPLE: 2

Write the symbolized form of the statement. If either Ram takes C++ or Kumar takes Pascal, then Latha will take Lotus.

R: Ram takes C++

K: Kumar takes Pascal

L: Latha takes Lotus

Solution:

Symbolic form: $(R \vee K) \rightarrow L$

Example 3

Let p, q, r represent the following propositions,

P: It is raining

q: The sun is shining

r: There are clouds in the sky

Symbolize the following statements.

- (1) If it is raining, then there are clouds in the sky
- (2) If it is not raining, then the sun is not shining and there are clouds in the sky.
- (3) The sun is shining if and only if it is not raining.

Solution:

Symbolic form:

- (1) $p \rightarrow r$
- (2) $\neg p \rightarrow (\neg q \wedge r)$
- (3) $q \leftrightarrow \neg r$

Example: 4

Symbolize the following statements:

- (1) If the moon is out and it is not snowing, then Ram goes out for a walk.
- (2) If the moon is out, then if it is not snowing, Ram goes out for a walk.
- (3) It is not the case that Ram goes out for a walk if and only if it is not snowing or the moon is out.

Solution:

Let the propositions be,

P: The moon is out

Q: It is snowing

R: Ram goes out for a walk.

Symbolic form:

- (1) $(p \wedge \neg q) \rightarrow r$
- (2) $p \rightarrow (\neg q \rightarrow r)$
- (3) $\neg(r \leftrightarrow (\neg q \vee p))$

Example: 5

Symbolize the following using the propositions.

P: I finish writing my computer program before lunch

q: I shall play Tennis in afternoon.

r: The sun is shining

s: The boundary is low.

- (1) If the sun is shining, I shall play tennis in the afternoon.
- (2) Finishing the writing of my computer program before lunch is necessary for playing tennis in this afternoon.
- (3) Low boundary and sunshine are sufficient to play Tennis in this afternoon.

Solution:

Symbolic form:

- (1) $r \rightarrow q$
- (2) $q \rightarrow p$
- (3) $(s \wedge r) \rightarrow q$

Construction of Truth Tables

EXAMPLE: 1

Show that the truth values of the formula $P \wedge (P \rightarrow Q) \rightarrow Q$ are independent of their components.

Solution:

The truth table for the formula is,

P	Q	$P \rightarrow Q$	$P \wedge (P \rightarrow Q)$	$(P \wedge (P \rightarrow Q)) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The truth values of the given formula are all true for every possible truth values of P and Q. Therefore, the truth value of the given formula is independent of their components.

Example 1. Without constructing the truth table show that

$$p \rightarrow (q \rightarrow p) \equiv \neg p(p \rightarrow q)$$

Solution

$$\begin{aligned} p \rightarrow (q \rightarrow p) &\equiv p \rightarrow (\neg q \vee p) \\ &\equiv \neg p \vee (\neg q \vee p) \\ &\equiv \neg p \vee (p \vee \neg q) \end{aligned}$$

$$\equiv (\neg p \vee p) \vee \neg q$$

$$\equiv T \vee \neg q$$

$$\equiv T.$$

Example 2. Prove that $p \rightarrow q$ is logically prove that $(\neg p \vee q)$

Solution:

p	q	$p \rightarrow q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

EXAMPLE: 2

Write the symbolized form of the statement. If either Ram takes C++ or Kumar takes pascal, then Latha will take Lotus.

R:Ram takes C++

K:Kumar takes Pascal

L:Latha takes Lotus

Solution:

Symbolic form: $(R \vee K) \rightarrow L$

Tautology.

A statement that is true for all possible values of its propositional variables is called

a tautology universally valid formula or a logical truth.

Example:1. Write the converse, inverse, contra positive of 'If you work hard then you will be rewarded'

Solution:

p: you will be work hard.

q: you will be rewarded.

$\neg p$: You will not be work hard.

$\neg q$: You will no tbe rewarded.

Converse: $q \rightarrow p$, If you will be rewarded then you will be work hard

Contrapositive: $\neg q \rightarrow p$, if You will not be rewarded then You will not be work hard.

Inverse: $\neg p \rightarrow \neg q$, if You will not be work hard then You will no tbe rewarded.

Example:2. Write the converse, inverse, contra positive of 'If you work hard then you will be rewarded'

Solution:

p : you will be work hard.
 q : you will be rewarded.
 $\neg p$: You will not be work hard.
 $\neg q$: You will not be rewarded.

Converse: $q \rightarrow p$, If you will be rewarded then you will be work hard

Contrapositive: $\neg q \rightarrow \neg p$, if You will not be rewarded then You will not be work hard.

Inverse: $\neg p \rightarrow \neg q$, if You will not be work hard then You will not be rewarded.

Example 4. Prove that $(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$

Proof:

Let $S: (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$

To prove: S is a tautology

P	Q	R	$(P \rightarrow Q)$	$(Q \rightarrow R)$	$(P \rightarrow R)$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$	S
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The last column shows that S is a tautology

3.3 PROPOSITIONAL EQUIVALENCE:

Logical Equivalence:

Let p and q be two statements formulas, p is said to be logically equivalent to q if $p \wedge q$ have the same set of truth values or equivalently $p \wedge q$ are logically equivalent if $p \leftrightarrow q$ is tautology.

Hence, $p \leftrightarrow q$ if and only if $p \leftrightarrow q$ is a tautology.

Logical Implication or Tautological Implication

A statement formula A logically implies another, statement formula B if and only if $A \rightarrow B$ is a tautology.

$\therefore A \Rightarrow B$ [A logically iff $A \rightarrow B$ is tautology, implies B]

If $A \Rightarrow B$, then

A is called antecedent

B is called consequent.

Further $A \Rightarrow B$ guarantees that B has the truth value T whenever A has the truth value T.

∴ In order to show any of the given implications, it is sufficient to show that an assignment of the truth value T to the antecedent of the given conditional leads to the truth value T for the consequent.

1. Prove without using truth table $(P \rightarrow Q) \wedge \neg Q \Rightarrow \neg P$

Proof:

Antecedent: $(P \rightarrow Q) \wedge \neg Q$

Consequent: $\neg P$

Assume that, the antecedent has the truth value T.

∴ $\neg Q$ And $(P \rightarrow Q)$ both are true.

⇒ Truth value of Q is F and the truth value of P is also F.

∴ Consequent $\neg P$ is true.

∴ The truth of the antecedent implies the truth of the consequent.

∴ $(P \rightarrow Q) \wedge \neg Q \Rightarrow \neg P$

Example:1 Without constructing the truth table show that $p \rightarrow (q \rightarrow p) \equiv \neg p(p \rightarrow q)$

Solution

$$\begin{aligned} p \rightarrow (q \rightarrow p) &\equiv p \rightarrow (\neg q \vee p) \\ &\equiv \neg p \vee (\neg q \vee p) \\ &\equiv \neg p \vee (p \vee \neg q) \\ &\equiv (\neg p \vee p) \vee \neg q \\ &\equiv T \vee \neg q \\ &\equiv T. \end{aligned}$$

Example 2: Show that $\neg(p \leftrightarrow q) \equiv (p \vee q) \wedge \neg(p \wedge q)$ without constructing the truth table

Solution :

$$\begin{aligned} \neg(p \leftrightarrow q) &\equiv (p \vee q) \wedge \neg(p \wedge q) \\ \neg(p \leftrightarrow q) &\equiv \neg(p \rightarrow q) \wedge (q \rightarrow p) \\ &\equiv \neg(\neg p \vee q) \wedge (\neg q \vee p) \\ &\equiv \neg(\neg p \vee q) \wedge \neg q \vee ((\neg p \vee q) \wedge p) \end{aligned}$$

$$\begin{aligned}
&\equiv \neg(\neg p \wedge \neg q) \vee (q \wedge \neg q) \vee ((\neg p \wedge p) \vee (q \wedge p)) \\
&\equiv \neg(\neg p \vee q) \vee F \vee F \vee (q \wedge p) \\
&\equiv \neg(\neg p \vee q) \vee (q \wedge p) \\
&\equiv (p \vee q) \wedge (q \wedge p).
\end{aligned}$$

Consider $(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R) \Rightarrow \neg(P \vee Q) \vee \neg(P \vee R) \Rightarrow \neg((P \vee Q) \wedge (P \vee R)) \quad (2)$

Using (1) and (2)

$$((P \vee Q) \wedge (P \vee Q) \wedge (P \vee R)) \vee \neg((P \vee Q) \wedge (P \vee R))$$

$$\Rightarrow [(P \vee Q) \wedge (P \vee R)] \vee \neg[(P \vee Q) \wedge (P \vee R)] \Rightarrow T$$

Prove the following equivalences by proving the equivalences of the dual

$$\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$$

Solution: It's dual is

$$\neg((\neg P \vee Q) \wedge (\neg P \vee \neg Q)) \wedge (P \vee Q) \equiv P$$

Consider,

	Reasons
$\neg((\neg P \vee Q) \wedge (\neg P \vee \neg Q)) \wedge (P \vee Q) \equiv P$	
$\Rightarrow ((P \wedge \neg Q) \vee (P \wedge Q)) \wedge (P \vee Q)$	(Demorgan's law)
	(Commutative law)
$\Rightarrow ((Q \wedge P) \vee (\neg Q \wedge P)) \wedge (P \vee Q)$	(Distributive law)
	$(P \vee \neg P \Rightarrow T)$
$\Rightarrow ((Q \vee \neg Q) \wedge P) \wedge (P \vee Q)$	$(P \wedge T = P)$
$\Rightarrow (T \wedge P) \wedge (P \vee Q)$	(Absorption law)

Obtain DNF of $Q \vee (P \wedge R) \wedge \neg((P \vee R) \wedge Q)$.

Solution:

$$Q \vee (P \wedge R) \wedge \neg((P \vee R) \wedge Q)$$

$$\Leftrightarrow (Q \vee (P \wedge R)) \wedge \neg((P \vee R) \wedge Q) \quad (\text{Demorgan law})$$

$$\begin{aligned}
&\Leftrightarrow (Q \vee (P \wedge R)) \wedge ((\neg P \wedge \neg R) \vee \neg Q) \quad (\text{De Morgan law}) \\
&\Leftrightarrow (Q \wedge (\neg P \wedge \neg R)) \vee (Q \wedge \neg Q) \vee ((P \wedge R) \wedge \neg P \wedge \neg R) \vee ((P \wedge R) \wedge \neg Q) \\
&\quad (\text{Extended distributed law}) \\
&\Leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee F \vee (F \wedge R \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \quad (\text{Negation law}) \\
&\Leftrightarrow (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \quad (\text{Negation law})
\end{aligned}$$

Obtain Pcnf and Pdnf of the formula $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

Solution:

Let $S = (\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$P \leftrightarrow \neg Q$	S	Minterm	Maxterm
T	T	F	F	F	F	T	$P \wedge Q$	
T	F	F	T	T	T	T	$P \wedge \neg Q$	
F	T	T	F	T	T	T	$\neg P \wedge Q$	
F	F	T	T	T	F	F		$P \vee Q$

PCNF: $P \vee Q$ and PDNF: $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

$$P \rightarrow (P \wedge (Q \rightarrow P)).$$

Obtain PDNF of

Solution:

$$P \rightarrow (P \wedge (Q \rightarrow P)) \Leftrightarrow \sim P \vee (P \wedge (\sim Q \vee P))$$

$$\Leftrightarrow \sim P \vee (P \wedge \sim Q) \vee (P \wedge P)$$

$$\Leftrightarrow (\sim P \wedge T) \vee (P \wedge \sim Q) \vee (P \wedge P)$$

$$\Leftrightarrow (\sim P \wedge (Q \vee \sim Q)) \vee (P \wedge \sim Q) \vee (P \wedge (Q \vee \sim Q))$$

$$\begin{aligned}
&\Leftrightarrow (\sim P \wedge (Q \vee \sim Q)) \vee (P \wedge \sim Q) \vee (P \wedge (Q \vee \sim Q)) \\
&\Leftrightarrow (\sim P \wedge (Q \vee \sim Q)) \vee (P \wedge \sim Q) \vee (P \wedge (Q \vee \sim Q))
\end{aligned}$$

$$\begin{aligned}
& P \wedge Q \vee (\sim \\
& \quad P \wedge \sim Q) \vee \\
& \quad (P \wedge \sim Q) \\
& \quad \vee (P \wedge Q) \\
& P \wedge Q \vee (\sim \\
& \quad)
\end{aligned}$$

$$\begin{aligned}
& P \wedge \sim Q \vee \\
& (P \wedge \sim Q) \vee \\
& (P \wedge Q)
\end{aligned}$$

3.4 PREDICATES & QUANTIFIERS:

Quantifiers.

Universal Quantifiers:

The universal Quantification of $P(x)$ is the proposition." $P(x)$ is true for all values of x in the universe of discourse".

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$.here \forall is called the universal quantifier.

Existential Quantifier:

The existential Quantification of $P(x)$ is the proposition." There exists an element x in the universe of discourse such that $P(x)$ is true".

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.here \exists is called the existential quantifier.

Normal Forms:

DNF:

A formula which is equivalent to a given formula and which consists of sum of elementary products is called a disjunctive normal form of the given formula

PDNF: a formula which is equivalent to a given formula which consists of sum its minterms is called PDNF.

PCNF: a formula which is equivalent to a given formula which consists of product of maxterms is called PCNF.

Obtain PCNF of $(\neg p \rightarrow r) \wedge (q \leftrightarrow p)$. and hence obtain its PDNF.

Solution:

PCNF:

$$S \Leftrightarrow (\neg p \rightarrow r) \wedge (q \leftrightarrow p).$$

$$\Leftrightarrow (\neg p \rightarrow r) \wedge ((q \rightarrow p) \wedge (p \rightarrow q))$$

$$\begin{aligned}
&\Leftrightarrow (p \vee r) \wedge ((\neg q \vee p) \wedge (\neg p \vee q)) \\
&\Leftrightarrow ((p \vee r) \vee F) \wedge ((\neg q \vee p) \vee F) \wedge ((\neg p \vee q) \vee F) \\
&\Leftrightarrow ((p \vee r) \vee (q \wedge \neg q)) \wedge ((\neg q \vee p) \vee (r \wedge \neg r)) \wedge ((\neg p \vee q) \vee (p \wedge \neg p)) \\
&\Leftrightarrow ((p \vee r \vee q) \wedge (p \vee r \vee \neg q)) \wedge ((\neg q \vee p \vee r) \wedge (\neg q \vee p \vee \neg r)) \wedge \\
&\quad ((\neg p \vee q \vee r) \vee (\neg p \vee q \vee \neg r)) \\
&\Leftrightarrow ((p \vee r \vee q) \wedge ((\neg q \vee p \vee r) \wedge (\neg q \vee p \vee \neg r) \wedge ((\neg p \vee q \vee r) \vee (\neg p \vee \\
&\quad q \vee \neg r)))
\end{aligned}$$

PCNF of S: $((p \vee r \vee q) \wedge ((\neg q \vee p \vee r) \wedge (\neg q \vee p \vee \neg r) \wedge ((\neg p \vee q \vee r) \vee (\neg p \vee q \vee \neg r))$

PCNF of $\neg S$: $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

PDNI $\neg S \vdash (p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$.

3.5 RULES OF INFERENCE:

EXAMPLE:1 Verify that validating of the following inference. If one person is more successful than another, then he has worked harder to deserve success. Ram has not worked harder than Siva. Therefore, Ram is not more successful than Siva.

Solution:

Let the universe consists of all persons.

Let $S(x,y)$: x is more successful than y.

$H(x,y)$: x has worked harder than y to deserve success.

a: Ram

b: Siva

Then, given set of premises are

- 1) $(x)(y)[S(x,y) \rightarrow H(x,y)]$

2) $\neg H(a,b)$

3) Conclusion is $\neg S(a,b)$.

{1}	1) $(x)(y)[S(x,y) \rightarrow H(x,y)]$	Rule P
{2}	2) $(y)[S(a,y) \rightarrow H(a,y)]$	Rule US
{3}	3) $[S(a,b) \rightarrow H(a,b)]$	Rule US
{4}	4) $\neg H(a,b)$	Rule P
{5}	5) $\neg S(a,b)$	Rule T ($\neg P, P \rightarrow Q \Rightarrow \neg Q$)

EXAMPLE: 2 Show that $(x)(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$

Solution :

Steps	Premises	Rule	Reason
1	$(x)(H(x) \rightarrow M(x))$	P	Given premise
2	$H(s) \rightarrow M(s)$	US (1)	$(Vx)p(x) \Rightarrow p(y)$
3	$H(s)$	P	Given premise
4	$M(s)$	T	$(2)(3)(p \rightarrow q, p \Rightarrow q)$

EXAMPLE: 3 Show that $\neg p(a,b)$ follows logically from $(x)(y)(p(x,y) \rightarrow w(x,y))$ and $\neg w(a,b)$

Solution :

1. $(x)(y)(p(x,y) \rightarrow w(x,y))$ p
2. $(y), p(a,y) \rightarrow w(a,y)$ US, (1)
3. $P(a,b) \rightarrow w(a,b)$ US (2)
4. $\neg w(a,b)$ p Given
5. $\neg p(a,b)$ T (3),(4), $(p \rightarrow Q) \wedge \neg Q \Rightarrow \neg p$

EXAMPLE:4.

Symbolise: For every x, there exists a y such that $x^2+y^2 \geq 100$

Solution :

$$(\forall x)(\exists y)(x^2+y^2 \geq 100)$$

Example: Let p, q, r be the following statements:

p: I will study discrete mathematics q: I will watch T.V.

r: I am in a good mood.

Write the following statements in terms of p, q, r and logical connectives. (1)
If I do not study and I watch T.V., then I am in good mood.

(2) If I am in good mood, then I will study or I will watch T.V.

(3) If I am not in good mood, then I will not watch T.V. or I will study.

$$(1) (\neg p \wedge q) \rightarrow r$$

$$(2) r \rightarrow (p \vee q)$$

$$(3) \neg r \rightarrow (\neg q \vee p)$$

3.6 Introduction to proofs & strategy

Method of proofs :

Trivial proof:

In an implication $p \rightarrow q$, if we can establish that q is true, then regardless of the truth value of p , the implication $p \rightarrow q$. So the construction of a trivial proof of $p \rightarrow q$ needs to show that the truth value of q is true.

Vacuous proof:

If the hypothesis p of an implication $p \rightarrow q$ is false, then $p \rightarrow q$ is true for any proposition q .

Prove that $\sqrt{2}$ is irrational.

Solution :

Suppose $\sqrt{2}$ is irrational.

$$\therefore \sqrt{2} = \frac{p}{q} \text{ for } p, q \in \mathbb{Z}, q \neq 0, p \text{ & } q \text{ have no common divisor.}$$

$$\therefore \frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2.$$

Since p^2 is an even integer, p is an even integer.

$\therefore p = 2m$ for some integer m .

$$\therefore (2m)^2 = 2q^2 \Rightarrow q^2 = 2m^2$$

Since q^2 is an even integer, q is an even integer.

$\therefore q = 2k$ for some integer k .

Thus p & q are even. Hence they have a common factor 2. Which is a contradiction to our assumption.

$\therefore \sqrt{2}$ is irrational.