Module-5
Graph Theory

Module 5

5 Graph Theory

5.1 Graph and Graph Models

DEFINITION: Graph:

A Graph G=(V,E,ϕ) consists of a non empty set v={v1,v2,.....} called the set of nodes (Points, Vertices) of the graph, E={e1,e2,...} is said to be the set of edges of the graph, and – is a mapping from the set of edges E to set off ordered or unordered pairs of elements of V.

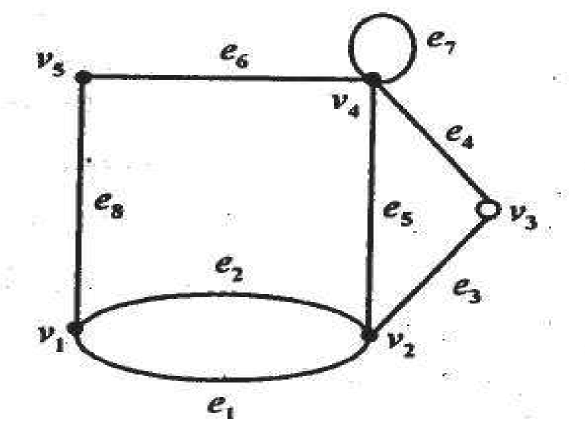
The vertices are represented by points and each edge is represented by a line diagrammatically.

DEFINITIONS:

From the figure we have the following definitions

V_1,v_2,v_3,v_4,v_5 are called vertices.

$e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8$ are called edges.



DEFINITION: Self Loop:

If there is an edge from v_i to v_i then that edge is called ***self loop*** or ***simply loop.***

For example, the edge e7 is called a self loop. Since the edge e_7 has the same vertex (v_4) as both its terminal vertices.

DEFINITION: Parallel Edges:

If two edges have same end points then the edges are called ***parallel edges.***

For example, the edge e_1 and e_2 are called parallel edges since e_1 and e_2 have the same pair of vertices (v_1,v_2) as their terminal vertices.

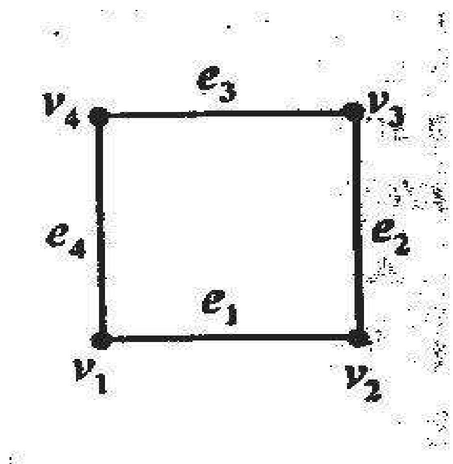
DEFINITION: Incident:

If the vertex v_i is an end vertex of some edge e_k and e_k is said to be ***incident*** with v_i.

DEFINITION: Adjacent edges and vertices:

Two edges are said to be adjacent if they are incident on a common vertex. In fig (i) the edges e_6 and e_8 are adjacent.

Two vertices v_i and v_j are said to adjacent if v_i v_j is an edge of the graph. (or equivalently (v_i,v_j) is an end vertices of the edge e_k)



For example, in fig., v_1 and v_5 are adjacent vertices.

DEFINITION: Simple Graph:

A graph which has neither self loops nor parallel edges is called a ***simple graph.***

<u>NOTE:</u> In this chapter, unless and otherwise stated we consider only simple undirected graphs.

DEFINITION: Isolated Vertex:

A vertex having no edge incident on it is called an ***Isolated vertex.*** It is obvious that for an isolated vertex degree is zero.

One can easily note that Isolated vertex is not adjacent to any vertex.

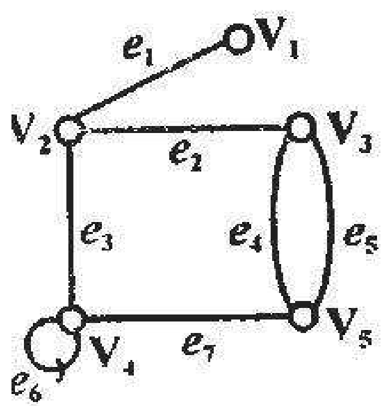
If fig (ii), v5 is isolated Vertex.

DEFINITION: Pendentant Vertex:

If the degree of any vertex is one, then that vertex is called pendent vertex.

EXAMPLE:

Consider the graph



In the above undirected graph

Vertices V={V_1, V_2, V_3, V_4, V_5}

Edges E={ e_1,e_2,....}

And e_1= < V_1, V_2> or < V_2,V_1>

e_2= < V_2,V_3> or < V_3,V_2>

e_4= < V_4,V_2> or < V_4,V_2>

e_5= < V_4,V_4>

In the above graph vertices V_1 and V_2, V_2 and V_3, V_3 and V_4, V_3 and V_5 are adjacent. Whereas V_1 and V_3, V_3 and V_4 are not adjacent.

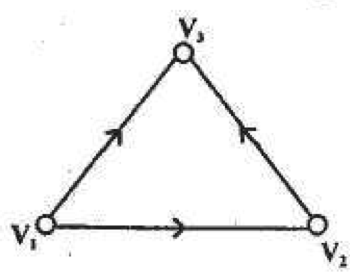
The edge e6 is called loop. The edges e_4 and e_5 are parallel edges.

Directed Edges:

In a graph G=(V,E), on edge which is associated with an ordered pair of V * V is called a ***directed edge*** of G.

If an edge which is associated with an unordered pair of nodes is called an ***undirected edge.***

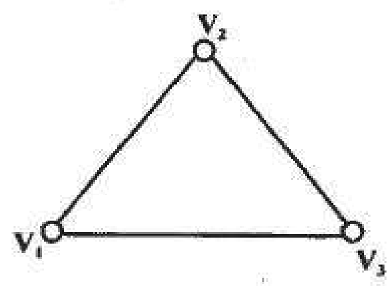
Digraph:



A graph in which every edge is directed edge is called a ***digraph*** or ***directed graph.***

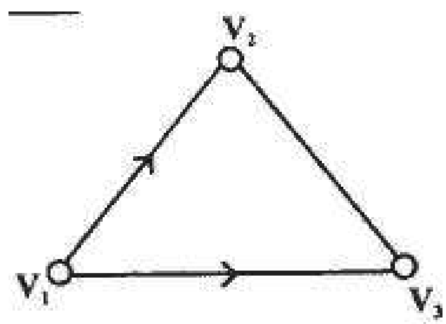
Undirected Graph:

A graph in which every edge is undirected edge is called an ***undirected graph.***



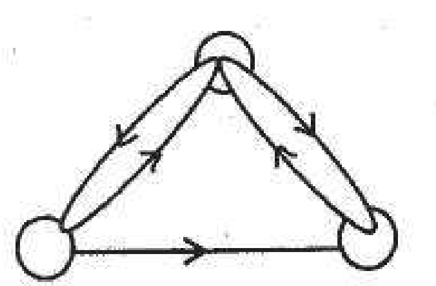
Mixed Graph:

If some edges are directed and some are undirected in a graph, the graph is called an ***mixedgraph.***



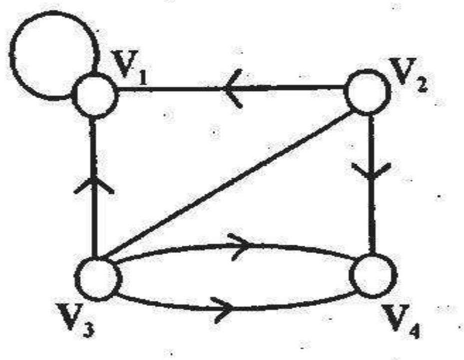
Multi Graph:

A graph which contains some parallel edges is called a ***multigraph.***



Pseudograph:

A graph in which loops and parallel edges are allowed is called a ***Pseudograph.***

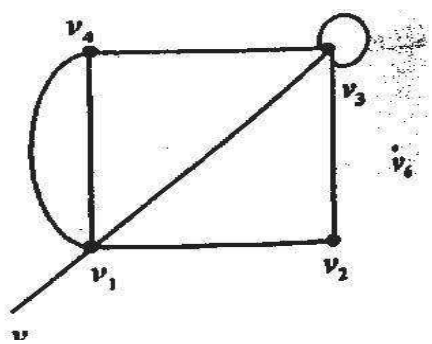


5.2 GRAPH TERMINOLOGY

DEFINITION: Degree of a Vertex:

The number of edges incident at the vertex v_i is called the **degree of the vertex** with self loops counted twice and it is denoted by d (v_i).

Example 1:



d (v_1) = 5 d (v_4) = 3

d (v_2) = 2 d (v_5) = 1

d (v_3) = 5 d (v_6) = 0

In-degree and out-degree of a directed graph:

In a directed graph, the in-degree of a vertex V, denoted by deg- (V) and defined by the number of edges with V as their terminal vertex.

The out-degree of V, denoted by deg+ (V), is the number of edges with V as their initial vertex.

NOTE: A loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

Theorem 1: (The Handshaking Theorem)

Let G= (V, E) be an undirected graph with 'e' edges. Then

deg(v)= 2e

The sum of degrees of all vertices of an undirected graph is twice the number of edges of the graph and hence even.

Proof:

Since every degree is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

Therefore, All the 'e' edges contribute (2e) to the sum of the degrees of vertices.

Therefore, **deg(v)= 2e**

Theorem 2:

In an undirected graph, the numbers of odd degree vertices are even.

Proof:

Let V_1 and V_2 be the set of all vertices of even degree and set of all vertices of odd degree, respectively, in a graph G= (V, E).

Therefore,

$$\mathbf{d(v)= \quad d(v_i)+ \; d(v_j)}$$

By handshaking theorem, we have

Since each deg (v_i) is even, is even.

As left hand side of equation (1) is even and the first expression on the RHS of (1) is even, we have the 2nd expression on the RHS must be even.

Since each deg (v_j) is odd, the number of terms contained in

i.e., The number of vertices of odd degree is even.

Theorem 3:

The maximum number of edges in a simple graph with 'n' vertices is n(n-1))/2.

Proof:

We prove this theorem by the principle of Mathematical Induction.

For n=1, a graph with one vertex has no edges.

Therefore, the result is true for n=1.

For n=2, a graph with 2 vertices may have at most one edge.

Therefore, 22-12=1

The result is true for n=2.

Assume that the result is true for n=k. i.e., a graph with k vertices has at most kk-12 edges.

When n=k+1. Let G be a graph having 'n' vertices and G' be the graph obtained from G by deleting one vertex say v ϵ V (G).

Since G' has k vertices, then by the hypothesis G' has at most kk-12 edges. Now add the vertex 'v' to G'. such that 'v' may be adjacent to all k vertices of G'.

Therefore, the total number of edges in G is,

Therefore, the result is true for n=k+1.

Hence the maximum number of edges in a simple graph with 'n' vertices is nn-12.

Theorem 4:

If all the vertices of an undirected graph are each of degree k, show that the number of edges of the graph is a multiple of k.

Proof:

Let 2n be the number of vertices of the given graph.

Let ne be the number of edges of the given graph.

By Handshaking theorem, we have

Therefore, the number of edges of the given graph is amultiple of k.

5.3 **SPECIAL TYPES OF GRAPHS**

Regular graph:

Definition: *Regular graph:*

If every vertex of a simple graph has the same degree, then the graph is called a *regular graph.*

If every vertex in a regular graph has degree k,then the graph is called *k-regular.*

DEFINITION : *Complete graph:*

In a graph, if there exist an edge between every pair of vertices,then such a graph is called complete graph.

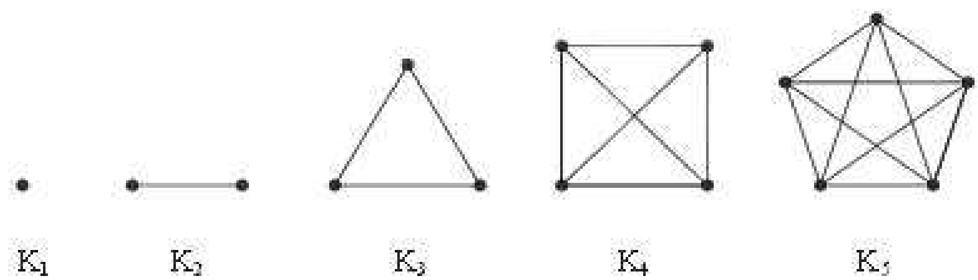


Fig. 1.10 Some complete graphs.

i.e., In a graph if every pair of vertices are adjacent,then such a graph is called complete graph.

If is noted that, every complete graphis a regular graph.In fact every complete graph with graph with n vertices is a (n-1)regular graph.

SUBGRAPH

A graph H =(V', E') is called a subgraph of G = (V, E), if V' **C V** and **E' C E.**

In other words, a graph H is said to be a subgraph of G if all the vertices and all edges of H are in G and if the adjacency is preserve in H exactly as in G.

Hence, we have the following:

(i) **Each graph has its own subgraph.**

(ii) **A single vertex in agraph G is a subgraph of G.**

(iii) **A single edge in G, together with its end vertices is also a subgraph of G.**

(iv) **A subgraph of a subgraph of G is also a subgraph of G.**

Note: Any sub graph of a graph G can be obtained by removing certain vertices and edges from G. It is to be noted that the removal of an edges does not go with the removal of its adjacent vertices, where as the removal of any edge incident on it.

Bipartite graph:

A graph G is said to be **bipartite** if its vertex set V (G) can be partitioned into two disjoint non empty sets V1 and V2, V1 U V2=V(G), such that every edge in E(G) has one end vertex in V1 and another end vertex in V2. (So that no edges in G, connects either two vertices in V1 or two vertices in V2.)

Examples of bipartite and complete bipartite graphs are shown in Figure 1.11.

(a) A bipartite graph. (b) A complete bipartite graph $K_{3,4}$.

Fig. 1.11 Two bipartite graphs.

Complete Bipartite Graph:

A bipartite graph G, with the bipartition V1 and V2, is called ***complete bipartite graph,*** if every vertex in V1 is adjacent to every vertex in V2.Clearly, every vertex in V2 is adjacent to every vertex in V1.

A complete bipartite graph with ‘m’ and ‘r’ vertices in the bipartition is denoted by km,n.

Incidence Matrix

Let G be a graph with n vertices, m edges and without self-loops. The incidence matrix A of G is an $n \times m$ matrix $A = [a_{ij}]$ whose n rows correspond to the n vertices and the m columns correspond to m edges such that

$$a_{ij} = \begin{cases} 1, & \textit{if jth edge } m_j \textit{ is incident on the ith vertex} \\ 0, & \textit{otherwise.} \end{cases}$$

It is also called *vertex-edge incidence matrix* and is denoted by $A(G)$.

Example Consider the graphs given in Figure 10.1. The incidence matrix of G_1 is

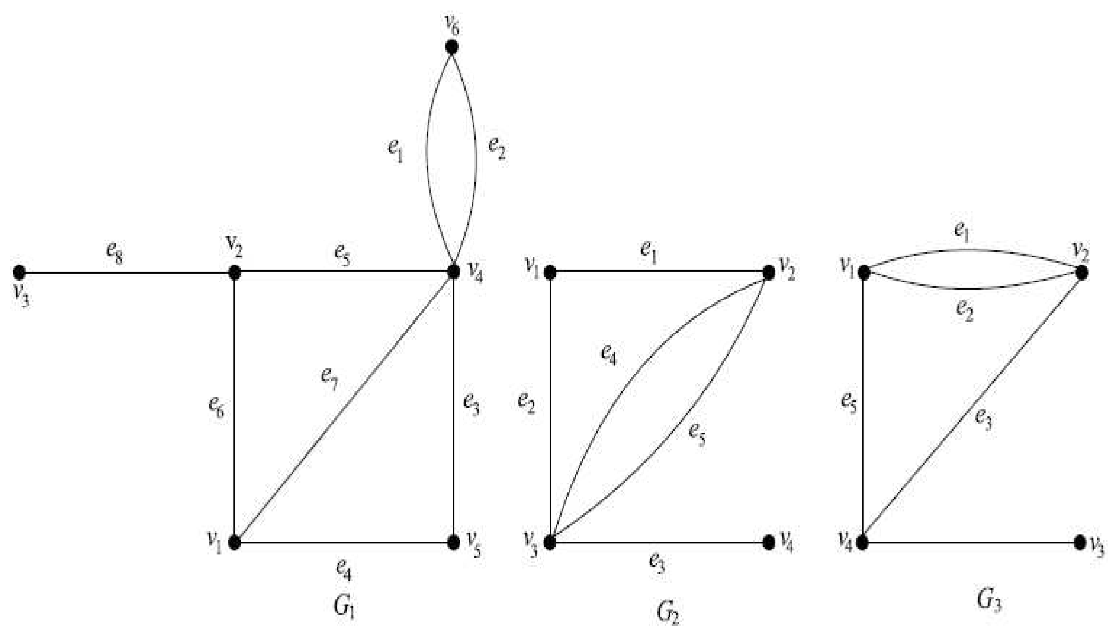
$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

The incidence matrix of G_2 is

$$A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

The incidence matrix of G_3 is

$$A(G_3) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$



(a) (b) (c)

The incidence matrix contains only two types of elements, 0 and 1. This clearly is a binary matrix or a (0, 1)-matrix.

We have the following observations about the incidence matrix A.

1. Since every edge is incident on exactly two vertices, each column of A has exactly two one's.

2. The number of one's in each row equals the degree of the corresponding vertex.

3. A row with all zeros represents an isolated vertex.

4. Parallel edges in a graph produce identical columns in its incidence matrix.

5. If a graph is disconnected and consists of two components G_1 and G_2, the incidence matrix $A(G)$ of graph G can be written in a block diagonal form as

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix},$$

   where $A(G_1)$ and $A(G_2)$ are the incidence matrices of components G_1 and G_2. This observation results from the fact that no edge in G_1 is incident on vertices of G_2 and vice versa. Obviously, this is also true for a disconnected graph with any number of components.

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

Path Matrix

Let G be a graph with m edges, and u and v be any two vertices in G. The path matrix for vertices u and v denoted by $P(u, v) = [p_{ij}]_{q \times m}$, where q is the number of different paths between u and v, is defined as

$$p_{ij} = \begin{cases} 1, & if\ jth\ edge\ lies\ in\ the\ ith\ path, \\ 0, & otherwise. \end{cases}$$

Clearly, a path matrix is defined for a particular pair of vertices, the rows in $P(u, v)$ correspond to different paths between u and v, and the columns correspond to different edges in G. For example, consider the graph in Figure 10.10.

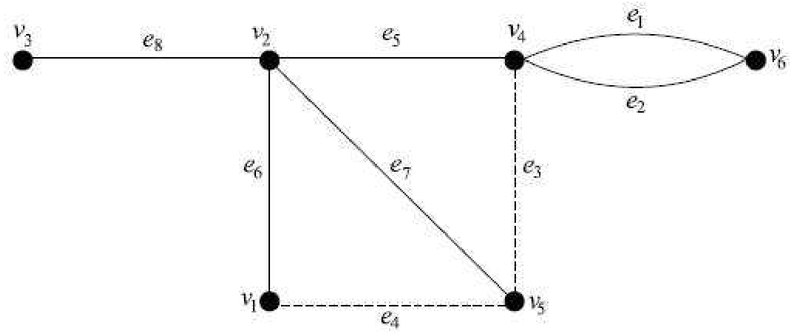


Fig. 10.10

The different paths between the vertices v_3 and v_4 are

$p_1 = \{e_8, e_5\}$, $p_2 = \{e_8, e_7, e_3\}$ and $p_3 = \{e_8, e_6, e_4, e_3\}$.

The path matrix for v_3, v_4 is given by

$$P(v_3, v_4) = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

We have the following observations about the path matrix.

1. A column of all zeros corresponds to an edge that does not lie in any path between u and v.
2. A column of all ones corresponds to an edge that lies in every path between u and v.
3. There is no row with all zeros.
4. The ring sum of any two rows in $P(u, v)$ corresponds to a cycle or an edge-disjoint union of cycles.

Adjacency Matrix

Let $V = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ and without parallel edges. The adjacency matrix of G is an $n \times n$ symmetric binary matrix $X = [x_{ij}]$ defined over the ring of integers such that

$$x_{ij} = \begin{cases} 1, & if\ v_i v_j \in E, \\ 0, & otherwise. \end{cases}$$

Example Consider the graph G given in Figure 10.12.

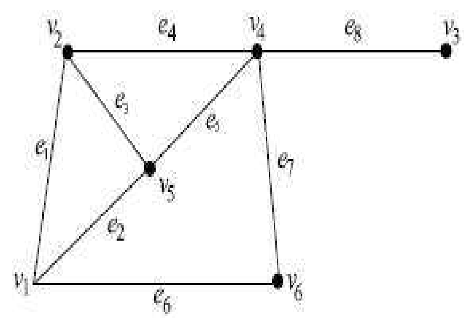


Fig. 10.12

The adjacency matrix of G is given by

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}.$$

We have the following observations about the adjacency matrix X of a graph G.

1. The entries along the principal diagonal of X are all zeros if and only if the graph has no self-loops. However, a self-loop at the ith vertex corresponds to $x_{ii} = 1$.
2. If the graph has no self-loops, the degree of a vertex equals the number of ones in the corresponding row or column of X.
3. Permutation of rows and the corresponding columns imply reordering the vertices. We note that the rows and columns are arranged in the same order. Therefore, when two rows are interchanged in X, the corresponding columns are also interchanged. Thus two graphs G_1 and G_2 without parallel edges are isomorphic if and only if their adjacency matrices $X(G_1)$ and $X(G_2)$ are related by

   $$X(G_2) = R^{-1}X(G_1)R,$$

   where R is a permutation matrix.
4. A graph G is disconnected having components G_1 and G_2 if and only if the adjacency matrix $X(G)$ is partitioned as

4. A graph G is disconnected having components G_1 and G_2 if and only if the adjacency matrix $X(G)$ is partitioned as

$$X(G) = \left[\begin{array}{ccc} X(G_1) & : & O \\ .. & : & .. \\ O & : & X(G_2) \end{array}\right],$$

where $X(G_1)$ and $X(G_2)$ are respectively the adjacency matrices of the components G_1 and G_2. Obviously, the above partitioning implies that there are no edges between vertices in G_1 and vertices in G_2.

5. If any square, symmetric and binary matrix Q of order n is given, then there exists a graph G with n vertices and without parallel edges whose adjacency matrix is Q.

GRAPH ISOMORPHISM

DEFINITION:

Two graphs G_1 and G_2 are said to be isomorphic to each other, if there exists a one-to-one correspondence between the vertex sets which preserves adjacency of the vertices.

Note: If G_1 and G_2 are isomorphic then G_1 and G_2 have,

- **(i) The same number of vertices.**
- **(ii) The same number of edges**
- **(iii) An equal number of vertices with a given degree.**

Note: However, these conditions are not sufficient for graph isomorphism.

ISOMORPHISM AND ADJACENCY:

RESULT 1:

Two graphs are isomorphic if and only if their vertices can be labeled in such a way that the corresponding adjacency matrices are equal.

RESULT 2:

Two simple graphs G1 and G2 are isomorphic if and only if their adjacency matrices A1 and A2 are related $A1=P^{-1} A_2 P$ where P is a permutation matrix.

Note:

A matrix whose-rows are the rows of the unit matrix but not necessarily in their natural order is called permutation matrix.

<u>Example:</u>
Test the Isomorphism of the graphs by considering the adjacency matrices.

Let A1 and. A2 be the adjacency matrices of G1 and G2 respectively.

$$A_1 = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Now $A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ (Interchaning Column 3 and Column 4)

$\sim \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ (Interchaning Row 3 and Row 4)

$\sim A_2$

Since A_1 and A_2 are similar, the corresponding graphs G_1 and G_2 are Isomorphic.

Paths,Reachability and Connectedness:

DEFINTTIONS:

Path:

A Path in a graphi s a sequence v1,v2,v3……vk of vertices each adjacent to the next.ln other words,starting with the vertex v1 one can travel along edges(v1,v2),(v2,v3)..and reach the vertex vk.

Length of the path:

The number of edges appearingi n the sequence of a path is called the length of Path.

Cycle or Circuit:

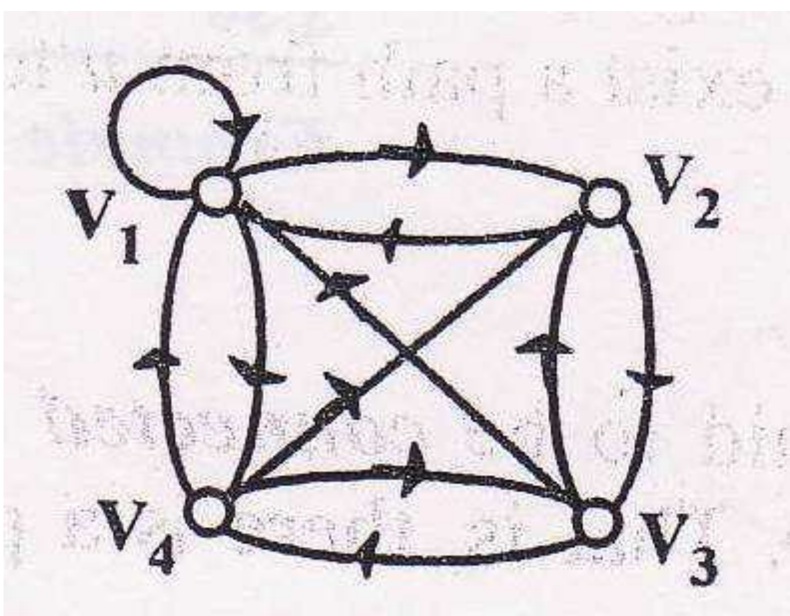
A path which originates and ends in the same node is called a cycle of circuit.

A path is said to be simple if all the edges in the path are distinct.

A path in which all the vertices are traversed only once is called an elmentary Path.

Example I :

Consider the graph:



Then some of the paths originating in node V1 and ending in node v1 are:

P1=(<V1 ,V2> ,<V2V3>)

P2=(<V1 ,V4> ,<V4 ,V3>)

P3 = (<V1 ,V2>,(V2, V4), <V4, V3>)

P4 = (<V1;V2>,<V2,V4>,<V4,V1>,<V1,V2>, <V2,V3>)

P5 = (<V1,V2>,<V2,V4>,<V4,V1 >,<V1,V4>,<V4,V3>)

P6= (<V1,Vl>, (V1,V1), (V1,V2), < V2, V3>)

Here,paths P1P2 and P3 are elementary path.

Path P5 is simple but not elementary.

DEFINITION:

REACHABLE:

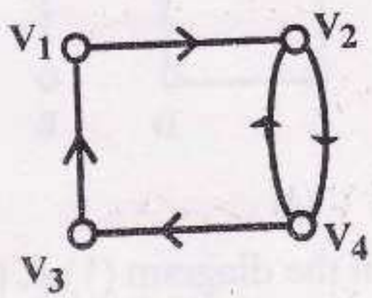
A node v of a simple digraph is-said to ber eachable from the node u of the same graph, if there exist a path from u to v.

Connected Graph :

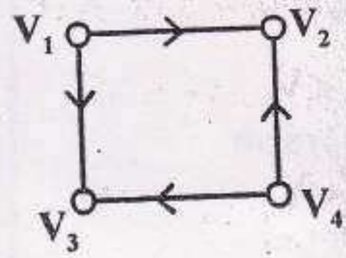
An directed graph is said to be connected if any pair of nodes are reachable from one another that is, there is a path between any pair of nodes.

A graph which is not connected is called disconnected graph.

Example 1 :



Connected Graph



Not Connected Graph

Components of a Graph :

The connected subgraphs of a graph G are called components of the.' graph G.

Theorem :

A simple graph with 'n' vertices and 'k' components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges

Proof :

Let $n_1, n_2, \ldots\ldots, n_k$ be the number of vertices in each of k components of the graph G.

Then $n_1 + n_2 + \ldots\ldots + n_k = n = |V(G)|$

$$\sum_{i=1}^{k} n_i = n \qquad \ldots (1)$$

Now, $$\sum_{i=1}^{k} (n_i - 1) = (n_1 - 1) + (n_2 - 1) + \ldots\ldots + (n_k - 1)$$

$$= \sum_{i=1}^{k} n_i - k$$

$$\sum_{i=1}^{k} (n_i - 1) = n - k$$

Squaring on both sides

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2$$

$$(n_1 - 1)^2 + (n_2 - 1)^2 + \ldots\ldots + (n_k - 1)^2 \leq n^2 + k^2 - 2nk$$

$$n_1^2 + 1 - 2n_1 + n_2^2 + 1 - 2n_2 + \ldots + n_k^2 + 1 - 2n_k \leq n^2 + k^2 - 2nk$$

DEFINITION:

Unilaterally Connected:

A simple digraph is said to be unilaterally connected if for any pair of nodes of the graph atleast one of the node of the pair is reachable from the node.

Strongly Connected:

A simple digraph is said to be strongly connected if for any pair of nodes of the graph both the nodes of the pair are reachable from the one another.

Weakly Connected:

We call a digraph is weakly.connected if it is connected.as an undirected graph in which the direction of the edges is neglected.

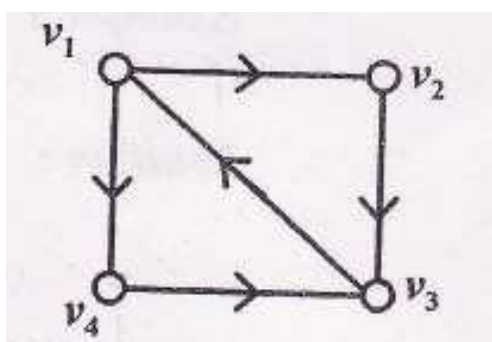
Note:

1.A unilateraaly connected digraph is weakly connectedbut a weakly connected digraph is not necessarily unilaterally connected.

2.A strongly connected digraph is both unilaterally and weakly connected.

EXAMPLE:

For example consider the graph:



It is strongly connected graph.

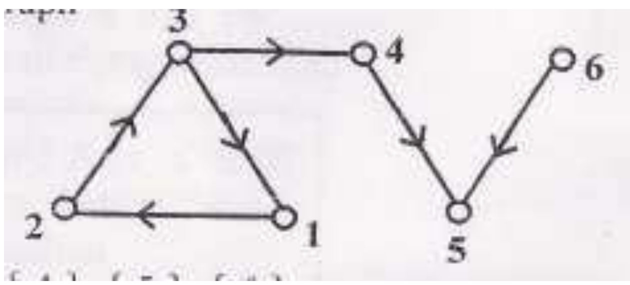
For,

The possible pairs of vertices of the graph are (v1 v2), (v1 v3), (v1 V4), (V2 V3) and (v2 V4)

(1) Consider the pair (v1 v2)

Then there is a path from v1 to v2,via v1-> v2 and path from v2-> v1,via v2->v3->v1
(2) Consider the pair (v1 v3)
There is a path from v1 to v3, via v1 -> v2-> v3 and path from v3 to v1 via v3 -> v1.
similarly we can prove it for the remaining pair of vertices,each vertices is reachable from other.
Given graph is strongly connected

DEFINITION:
For a simple digraph maximal strongly connected subgraph is called strong component.
For the digraph:



{1,2,3},{4},{5},{6} are strong component.

The possible Hamilton cycles are
(l) A-B-C-D-A
(2) A-D-C-B-A
(3) B->C-D-A-B
(4) B-A-D-C-B
(s) C-D-A-B-C
(6) C-B-A-D-C
(7) D-A-B-C-D
(S) D-C-B-A-D
(Since all the vertices appeares exactly once),but not all the edges.
Since,G 1 contains Hamiltonian cycle,G 1- is a Hamiltonian graph.
(2) G2 contains Hamiltonian paths,namely
(1) A-> B-C-D
(2) A -> B-D-C
(3) D->C-B-A etc.

We cannot find Hamiltonian cycle in G2.
Therefore G2 is not a Hamiltonian graph

Properties :
(1) A Hamiltonianc irbuitc ontainsa Hamiltonian path but a graph , Containing a Hamiltonian path need not have a Hamiltonian cycle.
(2) By deleting any one edge from Hamiltonian cycle,we can get Hamiltonian path.
(3) A graph may contain more than one Hamiltonian cycle.
(4) A complete graph k_n, will always have a Hamiltonian cycle, when n>=3

Note :
We don't have simple necessary and sufficient criteria for the existence of Hamiltonian cycles. However, we have many theorems that give sufficient conditions for the existence of Hamiltonian cycles.
Also, certain properties can be used to show that a graph Has no Hamiltonian cycle.F or example a, graph with a vertex of degree one cannot have a Hamiltonian cycle, since in a Hamiltonian cycle each vertex is incident with two edges in the cycle.

3.4 EULER GRAPH & HAMILTON GRAPH:

Example:Explain Konisberg bridge problem.Repersent the problem by mean of graph.Does the problem have a solution?

Solution: There are two islands A and B formed by a river.They are connected to each other and to the river banks C and D by means of 7-bridges

The problem is to start from any one of the 4 land areas.A,B,C,D, walk across each bridge exactly once and return to the starting point.(without swimmimg across the river)

This problem is the famous Konisberg bridge problem.

When the situation is represented by a graph,with vertices representating the land areas the edges representing the bridges,the graph will be shown as fig:

Theorem:

In a simple digraph,G=(V,E) every node of the digraph lies in exactly one strong component.

Proof:

Let v € V(G) and S be the set of all those vertices of G which are mutually reachable with v.

The problem is to find whether there is an Eulerian circuit or cycle(i.e.a circuit containing every edge exactly once) in a graph.

Here, we can not find a Eulerian circuit.Hence,Konisberg bridge problem has no solution .

EULER GRAPH:

Definition: Euler path:

A path of a graph G is called an Eulerian path,if it contains each edge of the graph exactly once.

Eulerian Circuit or Eulerian Cycle:

A circuit or cycle of a graph G is called an Eulerian circuit or cycle,if it includes each of G exactly once.

(Here starting and ending vertex are same).

An Eulerian circuit or cycle should satisfies the following conditions.

(1)Starting and ending points(vertices) or same.

(2)Cycle should contain all the edges of the graph but exactly once.

Eulerian Graph or Euler Graph:

Any graph containing an Eulerian circuit or cycle is called an Eulerian graph.

Theorem:

A connected graph is Euler graph(contains Eulerian circuit) if and only if each of its vertices is of even degree.

Proof:

Let G be any graph having Eulerian circuit(cycle) and let "C" be an Eulerian circuit of G with origin(and terminus) vertex as u.Each time a vertex as an internal of C,then two of the edges incident with v are accounted for degree.

We get,for internal vertex v € (G)

d(v)=2+2*{number of times u occur inside V

=even degree.

Conversely, assume each of its vertices has an even degree.

Claim: G has an Eulerian circuit.Support not, i.e.,Assume G be a connected graph which is not having an Euler circuit with all vertices of even degree and less number of edges.That is ,any degree having less number of edges than G,then it has an Eulerian circuit.Since each vertex of G has degree atleast two,therefore G contains closed path.Let C be a closed path of maximum possible length in G.If C itself has all the edges of G,then C itself an Euler circuit in G.

By assumption,C is not an Euler circuit of G and G-E© has some component G' with |E(G')|>0.C has less number of egdes than g, therefore C itself ia an Eulerian,and C has all the vertices of even degtee,thus the connected graph G' also has all the vertices of even degree.Since |E(G')|< |E(G)|,therefore G' has an Euler circuit C'.Bec ause G is connected,there is vertex v in both C and C'. Now join C and C' and transverse all the edges of C and C' with commen vertex v,we get CC' is a closed path in G and E (C C') > E (C), which is not possible for the chioices of C.

G has an Eulerian circuit.

G is Euler graph.